Qualifying Exam, Fall 2022
Mathematics

* This is a closed-book test (with cheat sheets provided), and no calculator is allowed.

* Work THREE out of the four problems, and clarify which three you want graded.

I want problems #______, #______, and # ______ to be graded.

1. Obtain the Laplace Transform of the following functions

   (a)

$$f(t) = \int_0^t \tau^2 e^{3\tau} \sin(t-\tau)\, d\tau$$

   (b)

$$\begin{aligned} f(t) &= 0, \quad 0 \le t < \pi/2, \\ &= \cos(2t), \quad \pi/2 < t < \pi, \\ &= 0, \quad t > \pi \end{aligned}$$

2. Using the method of separation of variables, show that the solution to the 1D heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

in a heat conducting rod of length $L = \pi$ subject to the following boundary and initial conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \quad u(x,0) = \cos x$$

is given by

$$u(x,t) = e^{-c^2 t} \cos x.$$

It is sufficient to consider only the cases $k < 0$ and $k = 0$, but make sure to show clearly how the infinite series solution reduces to just one term.

3. Consider the following ODE, representing a driven linear spring-mass system, with mass m, spring constant $k > 0$, damping constant $c > 0$ and driving force $F(t)$. The displacement of the mass from its equilibrium position is denoted by x :

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

Let the parameters take on the following values,

$$m = 1, \quad c = 8, \quad k = 16,$$

(the units are not important) and let $F(t) = e^{-4t}$. If the initial dispalcement from the equilibrium position $x(0) = 0$ and initial velocity $x'(0) = -3$,

(a) obtain the unique solution for this initial value problem using any method known to you

(b) compute the time values at which the motion of the mass reverses direction (you may leave your answer in fractions since you do not have a calculator)

4. Consider the following function of a complex variable z:

$$f(z) = \frac{1}{z(z+1)}$$

(a) Expand $f(z)$ in a Laurent series about each of its singular points. Make sure in each case to determine the domain of convergence of the series and also sketch it.

(b) Compute the contour integrals $\oint_{C_1} f(z)\ dz$ and $\oint_{C_2} f(z)\ dz$, where C_1 is given by $\mid z \mid= 1$ and C_2 is given by $\mid z+1 \mid= 1$.

Useful Power Series:

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots, \quad \mid z \mid < 1$$

FORMULAS AND EQUATIONS

- **Laplace Transform Properties and Formulas.**

$$\mathcal{L}\left(af(t)+bg(t)\right) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t)),$$

$$\mathcal{L}\left(e^{at}f(t)\right) = F(s-a),$$

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s},$$

$$\mathcal{L}\left(e^{at}t^n\right) = \frac{n!}{(s-a)^{n+1}}, \quad n = 0, 1, 2, \cdots,$$

$$\mathcal{L}\left\{f(t-b)u(t-b)\right\} = e^{-bs}F(s),$$

$$\mathcal{L}\left\{\delta(t-b)\right\} = e^{-bs},$$

$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^nF}{ds^n},$$

$$\mathcal{L}\left\{f*g\right\} = \mathcal{L}(f)\mathcal{L}(g), \quad (f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

- **Cauchy-Riemann equations.**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- **Fourier Series.**

For a periodic function on the interval $a \le x < b$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right),$$

$$\begin{aligned}
a_0 &= \frac{1}{b-a}\int_a^b f(x)dx, \\
a_n &= \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx, \\
b_n &= \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2n\pi x}{b-a}\right)dx
\end{aligned}$$

- **Cauchy-Goursat Theorem.**

$$\oint_C f(z)dz = 0.$$

- **Cauchy's Integral Theorem.**

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$

- **Generalization of above formula for derivatives of all orders.**

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz, \quad n = 0, 1, 2, 3, \cdots$$

- **Residue Theorem.**

$$\oint_C f(z)\ dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z),$$

where $\text{Res}_{z=z_k} f(z)$ denotes the residue of $f(z)$ in the Laurent series with center at the singular point z_k lying within C.

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2 - a^2}$
5	t^a (a positive)	$\frac{\Gamma(a + 1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}$
6	e^{at}	$\frac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$