Qualifying Exam 2022

Select three problems

Dynamics

- **Problem 1: (10 points)**

Two particles with mass m are attached by a linear spring with a spring constant k and unstretched length $2r_0$, as shown in Figure 1. Assume that the system dynamics stays in a plane. There is no net external force on the system, so you can use the center of mass of the system is the origin of an inertial frame.

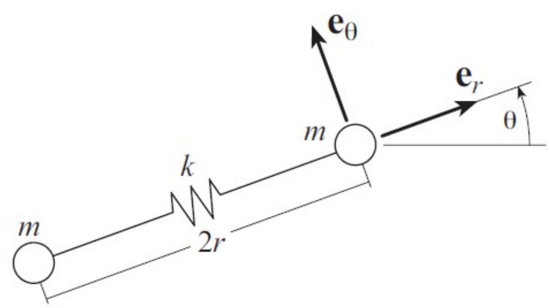


Figure 1

The goal is to obtain the possible motions of the system. Although not obvious at first, this system's dynamics can be reduced to a single ordinary differential equation.

1- Determine the second-order differential equations of motion whose solution would give $r(t)$ and $\theta(t)$ as functions of time; but do not solve them.
   (a) Using Lagrange's principle (2.5 points)
   (b) Using Newton's law (2.5 points)
2- Assume that the initial angular velocity is $\dot{\theta}_0$, the initial radial velocity is $\dot{r}_0$, and the initial separation $2r_0$ is the same as the unstretched length of the spring, determine an expression that relates $\dot{\theta}$ and r and the initial conditions. (2 points)
3- Using the conservation of energy, find a single scalar expression relating $\dot{r}$ and r of the form (3 points)

$$\dot{r}^2 = f(r)$$

- **Problem 2: (10 points)**

As shown in Figure 2, particles 1 and 2, each of mass m, are connected by a rigid massless rod of length l. Particle 1 can slide without friction along the floor while particle 2 can slide without friction on the wall. The dumbbell starts from rest with a lean angle θ=5° and falls under the influence of gravity.

(a) For what value of the angle θ will the dumbbell lose contact with the wall? (2 points)
(Hint: start with two generalized coordinates, x and θ, where x is the distance from the wall to particle 2, but impose constraint x = 0)

(b) For all that follows, consider the situation after the dumbbell loses contact with the wall (*i.e., x>0*). Find the 2nd-order equations of motion for x and θ. (3 points)

(c) Use the Routhian procedure to find a 2nd-order equation of motion for just the angle θ. (2 points)

(d) Using constants of motion and initial conditions, find a 1st-order equation of motion of the form: (3 points)

$$\dot{\theta} = f\ (\theta)$$

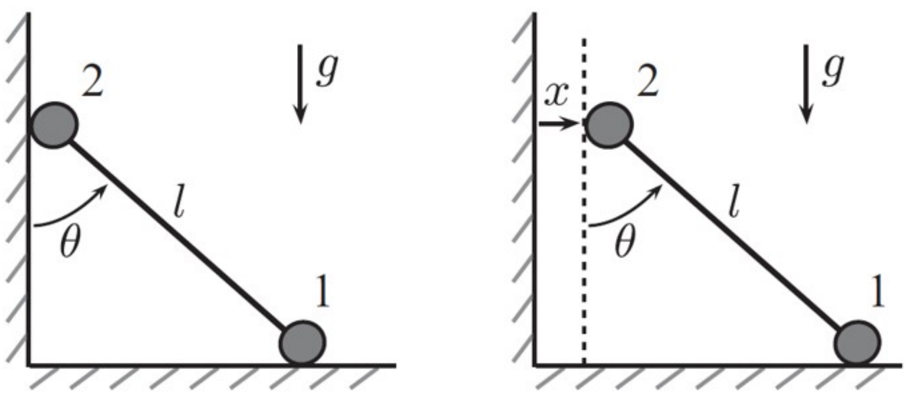


Figure 2

Vibrations

- **Problem 3: (10 points)**

A machine of mass M rests on a massless elastic floor, which is hinged at the wall, as shown in Figure 3(a). A shaker having total mass m_s and carrying two rotating unbalanced masses produces a vertical harmonic force $ml\omega^2 \sin \omega t$, where the frequency of rotation may be varied. The floor is made of steel (Young's modulus, E=200GPa) with the cross-section of b=40cm wide, h=2cm thick, and L=4m long; the total mass of the machine and shaker is M_{eq}=100kg; and the forcing magnitude is 500N when the operational speed of the shaker is at 300 rpm.

In the equivalent/simplified spring-damper-mass system (see Figure 3(b)), the equivalent mass (M_{eq}) and stiffness (K_{eq}) are, respectively, equal to: $K_{eq}=\frac{48EI}{L^3}$, $I=\frac{bh^3}{12}$, and $M_{eq}=m_s+M$.

1- Using Figure 3(b), determine the equation of motion using Newton's second law and the natural frequency. (2 points)

2- For an undamped system *(c=0),* what is the resulting steady-state amplitude of the machine-shaker setup due to rotating unbalance? (2 points)

3- For a damped system, the resonant amplitude of the system is X_r=3.41cm, determine the damping coefficient c of this system. (3 points)

4- Suppose an additional component is rigidly attached to the machine. This component adds an additional mass of 25kg. Determine the particular solution amplitude of oscillation for the undamped system with this additional mass. (3 points)

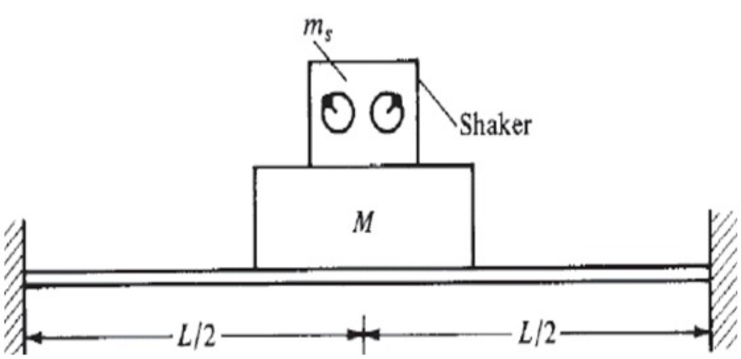


Figure 3(a)

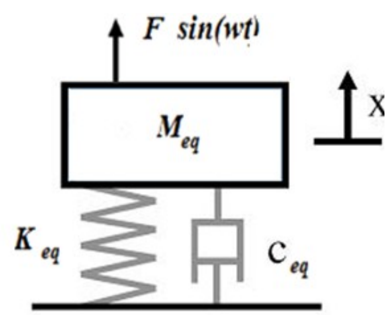


Figure 3(b)

- **Problem 4: (10 points)**

1- Considering l as the free length of the spring with a stiffness k**,** demonstrate that the equations of motion of a spring-pendulum system shown in Figure 4 can be expressed as: (3 points)

$$m\ddot{r} - m(l + r)\dot{\theta}^2 + kr - mg\cos(\theta) = 0$$

$$(l + r)\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin(\theta) = \frac{T(t)}{ml}$$

2- Determine the coupling terms and show their types (linear or nonlinear). Please justify. (1.5 points)

3- Assume small angles of oscillations, show that the linear form of the equations of motion can be expressed as: (1.5 points)

$$\begin{bmatrix} m & 0 \\ 0 & ml^2 \end{bmatrix} \begin{Bmatrix} \ddot{r} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mgl \end{bmatrix} \begin{Bmatrix} r \\ \theta \end{Bmatrix} = \begin{Bmatrix} mg \\ T(t) \end{Bmatrix}$$

4- Through a static analysis without any forcing, determine the equilibrium points (r_s, θ_s) of the linearized system. (2 points)

5- Determine the natural frequencies of the linearized system. (2 points)

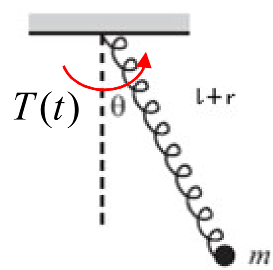


Figure 4

Formula Sheet

- Point P is fixed with the body

$\frac{d\overrightarrow{op}^{B}}{dt} = 0$, where: $\frac{d\vec{r}^{N}}{dt} = (\vec{\omega}_{\left(\frac{B}{N}\right)}\ X\ \vec{r})$

- P is moving w.r.t the body

$$\frac{d\vec{r}^{N}}{dt} = \frac{d\overrightarrow{op}^{B}}{dt} + (\vec{\omega}_{\left(\frac{B}{N}\right)}\ X\ \vec{r})$$

- Translating & Rotating frame

$$\vec{R} + \vec{e} = \vec{r}$$

$$\left(\overrightarrow{V^{P}}\right)_{N} = \left(\overrightarrow{V^{B}}\right)_{N} + \frac{d\vec{p}^{B}}{dt} + (\vec{\omega}_{\left(\frac{B}{N}\right)}\ X\ \vec{e})$$

- 5-part acceleration formula

$$\left(\overrightarrow{a^{P}}\right)_{N} = \left(\overrightarrow{a^{B}}\right)_{N} + \frac{d^{N}}{dt}\left(\frac{d\vec{e}^{B}}{dt} + (\vec{\omega}_{\left(\frac{B}{N}\right)}\ X\ \vec{e})\right)$$

$$\left(\overrightarrow{a^{P}}\right)_{N} = \left(\overrightarrow{a^{p}}\right)_{B} + \left(\overrightarrow{a^{B}}\right)_{N} + \left(2\vec{\omega}_{\left(\frac{B}{N}\right)}\ X\ \left(\overrightarrow{V_{p}}\right)_{B}\right) + \left(\vec{\omega}_{\left(\frac{B}{N}\right)}\ X\ \left(\vec{\omega}_{\left(\frac{B}{N}\right)}\ X\ \vec{e}\right)\right) + \left(\alpha_{\left(\frac{B}{N}\right)} X \vec{e}\right)$$

- Newton's 2nd Law

$$\sum \vec{F} = m\vec{a}$$

- Conservation of Energy

$$E = T + V \quad ; \quad E(t = t_1) = E(t = t_2)$$

$$\frac{d}{dt}(E) = 0$$

- Conservation of Angular Momentum

$$\overrightarrow{H_o} = m\vec{r} X\ \dot{\vec{r}}$$

$$\dot{\overrightarrow{H_o}} = m\vec{r} X\ \ddot{\vec{r}} = \overrightarrow{M_o}$$

$$\overrightarrow{H_p} = \sum_{i=1}^{N} \overrightarrow{\sigma_\iota} X m \dot{\overrightarrow{\sigma_\iota}}$$

$$\dot{\overrightarrow{H_p}} = \overrightarrow{M_p} + \sum_{i=1}^{N} m_i \ddot{\overrightarrow{r_p}} X \overrightarrow{\sigma_\iota}$$

- D’Alembert’s Principle

$\sum_{i=1}^{N} \left(\overrightarrow{f_{\iota a}} - m_i \ddot{\overrightarrow{r_\iota}}\right) \bullet \overrightarrow{B_{\iota J}} = 0$ For j=1,2,…,n

where $\overrightarrow{B_{\iota J}} = \frac{\partial \overrightarrow{r_\iota}}{\partial q_j}$

- Euler-Lagrange

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota}\right) - \frac{\partial L}{\partial q_i} = Q_{nc-i}$$

Formula sheet - Final -

- Constrained Version of Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_{nc\_i} + C_i$$

Kinetic and Conservative forces. — Q_{nc_i}: Nonconservative forces. — C_i: constraint forces.

$$\rightarrow C_i = \sum_{j=1}^{m} \lambda_j a_{ji} \quad / \; i = \{1, \dots n\}$$

where $a_{ji} = \frac{\partial \phi_j}{\partial q_i}$

λ_j : Lagrange multipliers.

m : number of constraints.

n : number of generalized coordinates.

$n - m$: number of degrees of freedom.

$$\rightarrow C_i = \sum_{k=1}^{N} \vec{f}_{ck} \cdot \vec{\beta}_{ki} \quad / \; i = \{1, \dots n\}$$

$$- \; Q_{nc\_i} = \sum_{k=1}^{N} \vec{f}_{ka} \cdot \vec{\beta}_{ki}$$

$\vec{f}_{ka}$ → applied forces.

$$\vec{\beta}_{ki} = \frac{\partial \vec{r}_k}{\partial q_i} = \frac{\partial \dot{\vec{r}}_k}{\partial \dot{q}_i}$$

* **Constraints**

→ Holonomic: Constraints are function of position $\phi_j(q) = 0 \quad / \; j = 1, \dots m$

→ Non-holonomic: Constraints cannot be put into above form.

Most of them can be put in linear non-holonomic form:

$$\sum_{i=1}^{n} a_{ji}(q)\, \dot{q}_i = 0 \; / \; j = 1, \dots m$$

m → number of constraints

→ find a_{ji}?

Routh method:

→ for handling "Ignorable coordinates".

$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \phi$ is an ignorable coordinate (ϕ → generalized coordinate)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \cancel{\frac{\partial L}{\partial \phi}}^{\,0} = Q_{nc\_\phi}$$

in case $Q_{nc-\phi} = 0$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{constat} = \beta_\phi$$

* Construct the Routhian function R:

$$R = L(\theta, \dot{\theta}, \dot{\phi}) - \beta_\phi \dot{\phi}$$

(ϕ → generalized coordinate)

→ Routh equation:

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0$$

→ looks like Lagrange's equation with $L \rightarrow R$

Formula Sheet

- **Euler-Lagrange equations:**

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_{nc,x}$$

- **SDOF spring-mass-damper system: free vibration**

$$\ddot{x} + 2\xi\omega_n\dot{x} + {\omega_n}^2 x = 0$$

$$\omega_n = \sqrt{\frac{k}{m}} \text{ and } \frac{c}{m} = 2\xi\omega_n$$

- Undamped free vibration ($\xi = 0$)

$$x(t) = Acos(\omega_n t) + Bsin(\omega_n t)$$

- Damped free vibration ($\xi \neq 0$)

Overdamped case: $\xi > 1$

$$\lambda_1 \text{ and } \lambda_2 \text{ are negative and real}$$

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Critically-damped case: $\xi = 1$

$$\lambda_1 \text{ and } \lambda_2 \text{ are equal (Repeated root)}$$

$$x(t) = (A + Bt)e^{-w_n t}$$

Underdamped case: $0 < \xi < 1$

$$\lambda_1 \text{ and } \lambda_2 \text{ are complex conjugates}$$

$$x(t) = e^{-\xi\omega_n t}[Acos(\omega_d t) + Bsin(\omega_d t)] = Xe^{-\xi\omega_n t}sin(\omega_d t + \phi)$$

where $X = \sqrt{x_o^2 + \frac{(v_o+\xi\omega_n x_o)^\wedge 2}{w_d^2}}$; $\phi = tan^{-1}\left(\frac{x_o\omega_d}{v_o+\xi\omega_n x_o}\right)$; $\omega_d = \omega_n\sqrt{1-\xi^2} = \frac{2\pi}{t_d}$

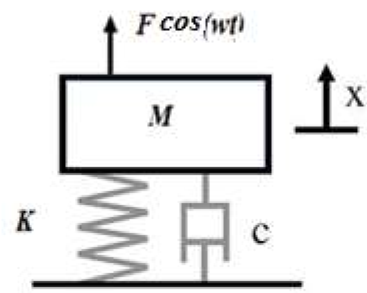
- Logarithmic decrement

$$\delta = \frac{1}{n-1}\ln\left(\frac{x_1}{x_n}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}}\,;\ \xi = \sqrt{\frac{1}{1+\left(\frac{2\pi}{\delta}\right)^2}}$$

- **Linearization**

$\cos(\theta) \approx 1$ and $\sin(\theta) \approx \theta$

- **Forced excitations**
- EoM



$$\ddot{x} + 2\xi\omega_n\dot{x} + {\omega_n}^2 x = F_0\cos(\omega t)$$

- Particular solution (general form)

$$x_p = X_1\cos(\omega t) + X_2\sin(\omega t) = X\sin(\omega t + \phi)$$

where

$X_1 = \frac{({\omega_n}^2-\omega^2)F_0}{({\omega_n}^2-\omega^2)^2+(2\xi\omega_n\omega)^2}$ and $X_2 = \frac{(2\xi\omega_n\omega)F_0}{({\omega_n}^2-\omega^2)^2+(2\xi\omega_n\omega)^2}$

$X = \sqrt{{X_1}^2 + {X_2}^2}$ and $\phi = tan^{-1}\left(\frac{X_1}{X_2}\right) = tan^{-1}\left(\frac{\omega_n^2-\omega^2}{2\xi\omega_n\omega}\right)$

- Amplitude of oscillations

$\frac{X}{\Delta_{st}} = \left(\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\right)$ where $r = \frac{\omega}{\omega_n}$ and $\Delta_{st} = \frac{F}{k}$

- Resonant frequency and amplitude

$r_{peak} = \frac{\omega_r}{\omega_n} = \sqrt{1-2\xi^2}$ and $\frac{X_r}{\Delta_{st}} = \frac{1}{2\xi\sqrt{1-\xi^2}}$

- Quality factor

$Q = \frac{\omega_r}{\Delta\omega} = \frac{f_r}{\Delta f}$ and $Q \approx \frac{1}{2\xi}$ for small ξ

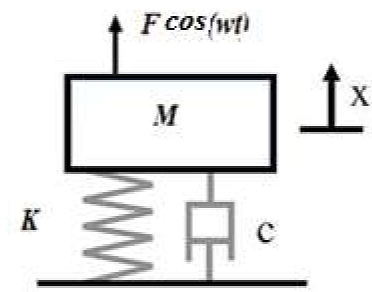
where $f_r = \frac{\omega_r}{2\pi}$

Formula Sheet

- $\omega_n = \sqrt{\frac{k}{m}}$
- $\frac{c}{m} = 2\xi\omega_n$
- $f_n = \frac{\omega_n}{2\pi}$

1- Forced excitations

- EoM



$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n{}^2 x = F_0\cos(\omega t)$$

- Particular solution (general form)

$$x_p = X_1\cos(\omega t) + X_2\sin(\omega t) = X\sin(\omega t + \phi)$$

where

$X_1 = \frac{(\omega_n{}^2-\omega^2)F_0}{(\omega_n{}^2-\omega^2)^2+(2\xi\omega_n\omega)^2}$ and $X_2 = \frac{(2\xi\omega_n\omega)F_0}{(\omega_n{}^2-\omega^2)^2+(2\xi\omega_n\omega)^2}$

$X = \sqrt{X_1{}^2 + X_2{}^2} = X = \frac{F_0}{\sqrt{(\omega_n{}^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}}$ and

$$\phi = tan^{-1}\left(\frac{X_1}{X_2}\right) = tan^{-1}\left(\frac{\omega_n^2-\omega^2}{2\xi\omega_n\omega}\right)$$

- Amplitude of oscillations

$$\frac{X}{\Delta_{st}} = \left(\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\right) \text{ where } r = \frac{\omega}{\omega_n} \text{ and } \Delta_{st} = \frac{F}{k} \text{ and } F_0 = \frac{F}{M}$$

- Resonant frequency and amplitude

$r_{peak} = \sqrt{1-2\xi^2}$ and $\frac{X_r}{\Delta_{st}} = \frac{1}{2\xi\sqrt{1-\xi^2}}$ and $\omega_r = \omega_n\sqrt{1-2\xi^2}$

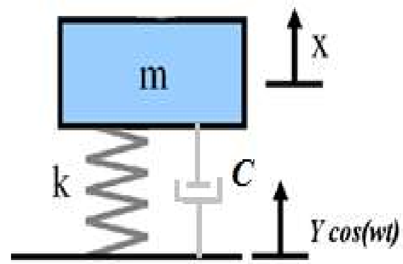
- Beating phenomenon ($\xi = 0 \Longrightarrow \omega_r = \omega_n$)

$$\sin\left(\alpha + \frac{\pi}{2}\right) = \cos(\alpha)$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

2 − Base excitations

- EoM



$$\ddot{x} + 2\xi\omega_n(\dot{x} - \dot{y}) + {\omega_n}^2(x - y) = 0$$

- $F_0 = Y\omega^2$
- Particular solution (general form)

$$z(t) = x(t) - y(t)$$

$$z(t) = Z_1\cos(\omega t) + Z_2\sin(\omega t) = Z\sin(\omega t + \Psi)$$

where

$\frac{Z_1}{Y} = \frac{(1-r^2)\,r^2}{(1-r^2)^2+(2\xi r)^2}$, $\frac{Z_2}{Y} = \frac{2\xi r^3}{(1-r^2)^2+(2\xi r)^2}$, and $\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$

$$\Psi = tan^{-1}\left(\frac{\omega_n^2 - \omega^2}{2\xi\omega_n\omega}\right)$$

- Amplitude of oscillations

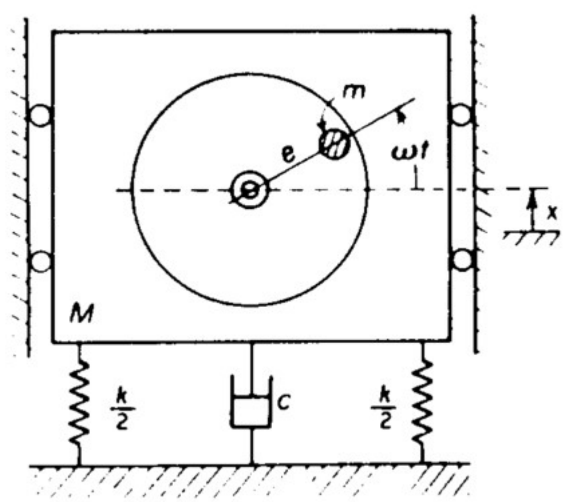
$$X = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}Y$$

- Resonant frequency

$$r_{peak} = \frac{\sqrt{\sqrt{1+8\xi^2}-1}}{2\xi}$$

3 − Rotating unbalance

- EoM



$$\ddot{x} + 2\xi\omega_n\dot{x} + {\omega_n}^2 x = \frac{m}{M}e\omega^2\sin(\omega t)$$

- $F_0 = \frac{F}{M} = \frac{me\ \ ^2}{M}$

- Centrifugal force

$$F_{centrifugal} = m\frac{v^2}{r}$$

- Amplitude of oscillations

$$X = \frac{me}{M}\frac{r^2}{\sqrt{(1-r^2)^2+(2\xi r)^2}} = \frac{me}{M}\frac{\omega^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\xi\omega_n\omega)^2}} = \frac{F_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\xi\omega_n\omega)^2}}$$

$$\psi = tan^{-1}\left(\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}\right) = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right)$$

- Resonant frequency and amplitude

$r_p = \frac{\omega_r}{\omega_n} = \frac{1}{\sqrt{1-2\xi^2}}$ and $\frac{MX_r}{me} = \frac{1}{2\xi\sqrt{1-\xi^2}}$

Formula sheet

1. Equations of Motion

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F(t)\}$$

- Euler-Lagrange equations of a system having $q_1, q_2 \dots \dots . . q_n$ as degrees of freedom (*L*=*T*-*V*)

$$\frac{d(\frac{\partial L}{\partial \dot{q}_i})}{dt} - \frac{\partial L}{\partial q_i} = Q_{nci}$$

- Law of Cosines

$$a^2 = b^2 + c^2 - 2bc * \cos(A)$$

2. Eigenvalue Problem (Free vibrations)

$$\{x(t)\} = \{X\}e^{j\omega t}$$

- $Substitute\ EVP\ into\ EoM$
- $Solve\ for\ \omega_1\ and\ \omega_2\ (natural\ frequencies)$
- $Solve\ for\ X_1\ and\ X_2\ (always\ set\ X_1 = 1)\ to\ find\ Modal\ Vectors$

3. General Solution/Normal Mode Vibrations

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (A\cos\omega_1 t + B\sin\omega_1 t)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\omega=\omega 1} + (C\cos\omega_2 t + D\sin\omega_2 t)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\omega=\omega 2}$$

4. Forced vibrations

$$[M]\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + [K]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A\cos(\Omega t + phi) \\ B\cos(\Omega t + phi) \end{pmatrix}$$

- In complex form:

$$[M]\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + [K]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Ae^{j(\Omega t + phi)} \\ Be^{j(\Omega t + phi)} \end{pmatrix}$$

- Steady-state solutions (X_1, X_2)
- $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \cos(\Omega t + phi)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{j(\Omega t + phi)} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\Omega^2 e^{j(\Omega t + phi)} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$(-\Omega^2 [M] + [K]) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{j(\Omega t + phi)} = \begin{pmatrix} A \\ B \end{pmatrix} e^{j(\Omega t + phi)}$$

$$(-\Omega^2 [M] + [K]) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$[A0] \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{pmatrix} A \\ B \end{pmatrix}$$

- $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \cos(\Omega t + phi) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$
- Inverse of a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\mathrm{ad} - \mathrm{bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$