Qualifying Exam 2019

Select three problems

Dynamics

- **Problem 1: (10 points)**

Particles 1 and 2, each of mass *m*, are connected by a rigid massless rod of length *l*, as shown in Figure 1. Particle 1 can slide without friction along the floor while particle 2 can slide without friction on the wall. The dumbbell starts from rest with a lean angle $\theta=5^\circ$ and falls under the influence of gravity.

1- For what value of the angle θ will the dumbbell lose contact with the wall? (2 points)

(Hint: start with two generalized coordinates, x and θ, where x is the distance from the wall to particle 2, but impose constraint $x = 0$)

2- For all that follows, consider the situation after the dumbbell loses contact with the wall (i.e., $x>0$). Find the 2nd-order equations of motion for x and θ. (3 points)

(c) Use the Routhian procedure to find a 2nd-order equation of motion for just the angle θ. (2 points)

(d) Using constants of motion and initial conditions, find a 1st-order equation of motion of the form: (3 points)

$$\dot{\theta} = f\ (\theta)$$

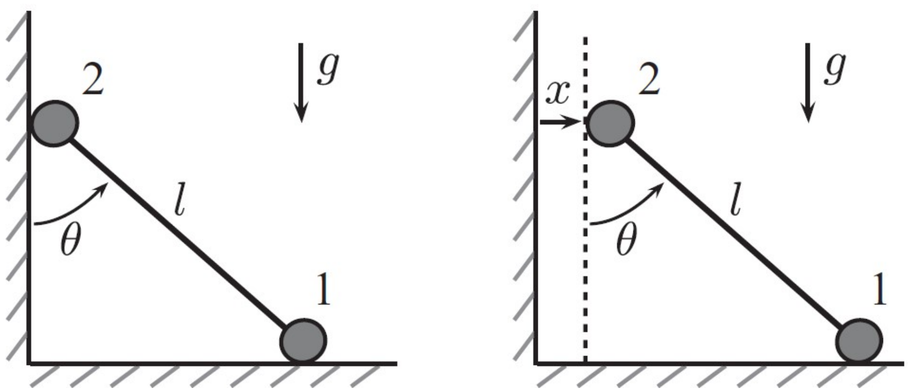


Figure 1

- **Problem 2: (10 points)**

Two masses, m_1 and m_2, are connected by an inextensible string: m_1 moves on the horizontal table, while m_2 hangs suspended by the string below the table. The string passes through a hole in the center of the table, as shown in Figure 2. You may assume that m_2 moves only in the vertical direction, and that neither m_1 nor m_2 pass through the hole.

1- Determine the expression of the total energy of this dynamical system (2 points)

2- Assume $\dot{r}(t = 0) = 0$ and apply the conservation of energy, determine a single scalar expression relating $\dot{r}$, $\dot{\theta}$, and r of the form: (3 points)

$$\dot{r}^2 = f(r,\dot{\theta})$$

where r is the distance from the hole to m_1. It should be mentioned that there will be constants in the function f related to physical constants and initial conditions.

3- Use the angular momentum to reduce the previous single scalar expression from $\dot{r}^2 = f(r,\dot{\theta})$ to $\dot{r}^2 = f(r)$ (3 points)

4- Using the conservation of energy method, demonstrate that this dynamical system is governed by the following equation of motion: (2 points)

$$2\ddot{r} - \frac{r_0^4\dot{\theta}_0^2}{r^3} + g = 0$$

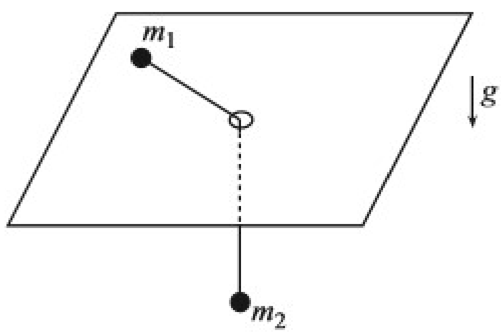


Figure 2

Vibrations

- **Problem 3: (10 points)**

The 2-DOF automobile model as shown in Figure 3 is considered. The mass of the automobile is M and the moment of inertia with respect of the center of mass is $J_o=Mr^2$.

1- Using Newton's second law and considering small oscillations, demonstrate that the general form of the equations of motion of the system can be expressed as: (2 points)

$$\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} \dfrac{2k_1+k_2}{M} & \dfrac{k_2l_2-2k_1l_1}{M} \\ \dfrac{k_2l_2-2k_1l_1}{J_0} & \dfrac{k_2l_2^2+2k_1l_1^2}{J_0} \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} \dfrac{F(t)}{M} \\ \dfrac{T(t)}{J_0} \end{Bmatrix}$$

2- Why the stiffness matrix is not symmetric? (0.5 points)

3- Assume the parameters of this dynamical system are M=350Kg, k_1=1000N/m, k_2=2400 N/m, l_1=1m, l_2=2m, and r=1m (the radius of gyration about cg).

(a) Determine the natural frequencies and corresponding mode shapes and depict the mode shapes. (2 points)

(b) Check the orthogonality of modes and then determine the normal mode vibrations of the system when the initial conditions are $x(0) = 1$, $\theta(0) = -1$, and zero initial velocities. (1.5 points)

4- Consider the general form of the equations of motion as presented in question 1 and assume that $F(t)=2F_0\cos(\Omega t)$ and $T(t)=T_0\cos(\Omega t)$:

(a) Determine the moment of inertia J_o that renders the displacement of the car stationary (i.e., $x(t) = 0$). (2 points)

(b) Determine the corresponding steady-state amplitude of the rotation. (2 points)

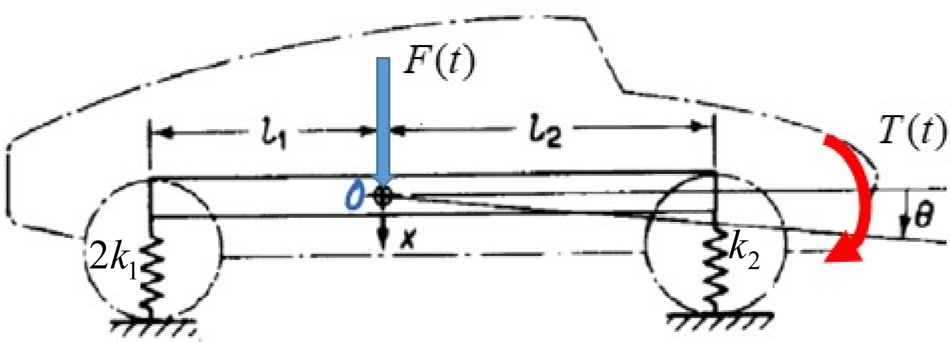


Figure 3

- **Problem 4: (10 points)**

1- Considering l as the free length of the spring with a stiffness k, demonstrate that the equations of motion of a spring-pendulum system shown in Figure 4 can be expressed as: (3 points)

$$m\ddot{r} - m(l+r)\dot{\theta}^2 + kr - mg\cos(\theta) = 0$$

$$(l+r)\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin(\theta) = \frac{T(t)}{ml}$$

2- Determine the coupling terms and show their types (linear or nonlinear). Please justify. (1.5 points)

3- Assume small angles of oscillations, show that the linear form of the equations of motion can be expressed as: (1.5 points)

$$\begin{bmatrix} m & 0 \\ 0 & ml^2 \end{bmatrix}\begin{Bmatrix} \ddot{r} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mgl \end{bmatrix}\begin{Bmatrix} r \\ \theta \end{Bmatrix} = \begin{Bmatrix} mg \\ T(t) \end{Bmatrix}$$

4- Through a static analysis without any forcing, determine the equilibrium points (r_s, θ_s) of the system. (2 points)

5- Determine the natural frequencies of the system. (2 points)

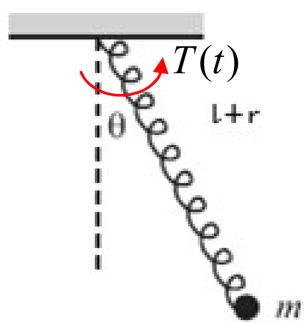


Figure 4

Formula Sheet

(QE 2019)

- Point P is fixed with the body

$\frac{d\overrightarrow{op}^B}{dt} = 0$, where: $\frac{d\vec{r}^N}{dt} = (\vec{\omega}_{\left(\frac{B}{N}\right)}\, X\, \vec{r})$

- P is moving w.r.t the body

$$\frac{d\vec{r}^N}{dt} = \frac{d\overrightarrow{op}^B}{dt} + (\vec{\omega}_{\left(\frac{B}{N}\right)}\, X\, \vec{r})$$

- Translating & Rotating frame

$\vec{R} + \vec{e} = \vec{r}$

$$\left(\overrightarrow{V^P}\right)_N = \left(\overrightarrow{V^B}\right)_N + \frac{d\vec{p}^B}{dt} + (\vec{\omega}_{\left(\frac{B}{N}\right)}\, X\, \vec{e})$$

- 5-part acceleration formula

$$\left(\overrightarrow{a^P}\right)_N = \left(\overrightarrow{a^B}\right)_N + \frac{d^N}{dt}\left(\frac{d\vec{e}^B}{dt} + (\vec{\omega}_{\left(\frac{B}{N}\right)}\, X\, \vec{e})\right)$$

$$\left(\overrightarrow{a^P}\right)_N = \left(\overrightarrow{a^p}\right)_B + \left(\overrightarrow{a^B}\right)_N + \left(2\vec{\omega}_{\left(\frac{B}{N}\right)}\, X\, \left(\overrightarrow{V_p}\right)_B\right) + \left(\vec{\omega}_{\left(\frac{B}{N}\right)}\, X\, \left(\vec{\omega}_{\left(\frac{B}{N}\right)}\, X\, \vec{e}\right)\right) + \left(\alpha_{\left(\frac{B}{N}\right)} X \vec{e}\right)$$

- Newton's 2nd Law

$$\sum \vec{F} = m\vec{a}$$

- Conservation of Energy

$E = T + V \quad ; \quad E(t = t_1) = E(t = t_2)$

$$\frac{d}{dt}(E) = 0$$

- Conservation of Angular Momentum

$\overrightarrow{H_o} = m\vec{r} X \dot{\vec{r}}$

$\dot{\overrightarrow{H_o}} = m\vec{r} X \ddot{\vec{r}} = \overrightarrow{M_o}$

$$\overrightarrow{H_p} = \sum_{i=1}^{N} \vec{\sigma_i} X m \dot{\vec{\sigma_i}}$$

$$\dot{\overrightarrow{H_p}} = \overrightarrow{M_p} + \sum_{i=1}^{N} m_i \ddot{\overrightarrow{r_p}} X \overrightarrow{\sigma_\iota}$$

- D'Alembert's Principle

$\sum_{i=1}^{N} \left(\overrightarrow{f_{\iota a}} - m_i \ddot{\vec{r_\iota}} \right) \bullet \overrightarrow{B_{\iota J}} = 0$ For j=1,2,…,n

where $\overrightarrow{B_{\iota J}} = \frac{\partial \vec{r_\iota}}{\partial q_j}$

- Euler-Lagrange

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota}\right) - \frac{\partial L}{\partial q_i} = Q_{nc-i}$$

Formula sheet - Final -

- Constrained Version of Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_{nc\_i} + C_i$$

Kinetic and conservative forces. / Q_{nc_i}: Nonconservative forces. / C_i: constraint forces.

$$\rightarrow C_i = \sum_{j=1}^{m} \lambda_j a_{ji} \quad / \quad i = \{1, \dots n\}$$

where $a_{ji} = \frac{\partial \phi_j}{\partial q_i}$

λ_j: Lagrange multipliers.

m: number of constraints.

n: number of generalized coordinates.

$n - m$: number of degrees of freedom.

$$\rightarrow C_i = \sum_{k=1}^{N} \vec{f}_{ck} \cdot \vec{\beta}_{ki} \quad / \quad i = \{1, \dots n\}$$

$$- Q_{nc\_i} = \sum_{k=1}^{N} \vec{f}_{ka} \cdot \vec{\beta}_{ki}$$

($\vec{f}_{ka}$: applied forces.)

$$\vec{\beta}_{ki} = \frac{\partial \vec{r}_k}{\partial q_i} = \frac{\partial \dot{\vec{r}}_k}{\partial \dot{q}_i}$$

* **Constraints**

→ Holonomic: constraints are function of position $\phi_j(q) = 0 \quad / \quad j = 1, \dots m$

→ Non-holonomic: Constraints cannot be put into above form.

Most of them can be put in linear non-holonomic form:

$$\sum_{i=1}^{n} a_{ji}(q)\, \dot{q}_i = 0 \quad / \quad j = 1, \dots m$$

(m: number of constraints)

↳ find a_{ji}?

Routh method:

↳ for handling "Ignorable coordinates".

$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \phi$ is an ignorable coordinate (ϕ: generalized coordinate)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = Q_{nc\_\phi}$$

($\frac{\partial L}{\partial \phi} \to 0$)

in case $Q_{nc\_\phi} = 0$

$$\hookrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{constat} = \beta_\phi$$

* Construct the Routhian function R:

$$R = L(\theta, \dot{\theta}, \dot{\phi}) - \beta_\phi \dot{\phi}$$

(ϕ: generalized coordinate)

→ Routh equation:

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0$$

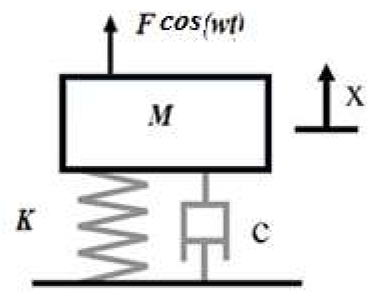
↳ looks like Lagrange's equation with $L \to R$

Formula Sheet

- $\omega_n = \sqrt{\frac{k}{m}}$
- $\frac{c}{m} = 2\xi\omega_n$
- $f_n = \frac{\omega_n}{2\pi}$

1- Forced excitations

- EoM



$$\ddot{x} + 2\xi\omega_n\dot{x} + {\omega_n}^2 x = F_0\cos(\omega t)$$

- Particular solution (general form)

$$x_p = X_1\cos(\omega t) + X_2\sin(\omega t) = X\sin(\omega t + \phi)$$

where

$$X_1 = \frac{({\omega_n}^2 - \omega^2)F_0}{({\omega_n}^2-\omega^2)^2 + (2\xi\omega_n\omega)^2} \text{ and } X_2 = \frac{(2\xi\omega_n\omega)F_0}{({\omega_n}^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}$$

$$X = \sqrt{{X_1}^2 + {X_2}^2} \text{ and } \phi = tan^{-1}\left(\frac{X_1}{X_2}\right)$$

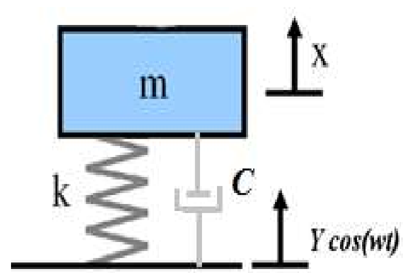
- Amplitude of oscillations

$$\frac{X}{\Delta_{st}} = \left(\frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}\right) \ where\ r = \frac{\omega}{\omega_n} \quad and\ \ \Delta_{st} = \frac{F}{k}\ and\ F_0 = \frac{F}{M}$$

- Resonant frequency and amplitude

$$r_{peak} = \sqrt{1-2\xi^2} \text{ and } \frac{X_r}{\Delta_{st}} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

2 − Base excitations



- EoM

$$\ddot{x} + 2\xi\omega_n(\dot{x} - \dot{y}) + {\omega_n}^2(x - y) = 0$$

- $F_0 = Y\omega^2$
- Particular solution (general form)

$$z(t) = x(t) - y(t)$$

$$z(t) = Z_1\cos(\omega t) + Z_2\sin(\omega t) = Z\sin(\omega t + \Psi)$$

where

$\frac{Z_1}{Y} = \frac{(1-r^2)+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}$ and $\frac{Z_2}{Y} = \frac{2\xi r^3}{(1-r^2)^2+(2\xi r)^2}$

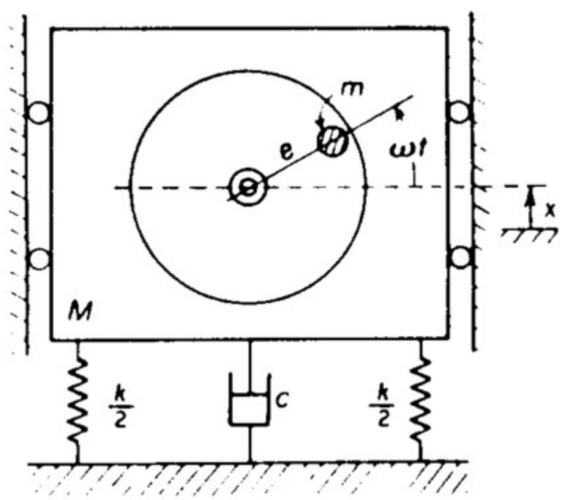
- Amplitude of oscillations

$$X = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}} Y$$

- Resonant frequency

$$r_{peak} = \frac{\sqrt{\sqrt{1+8\xi^2}-1}}{2\xi}$$

3 − Rotating unbalance



- EoM

$$\ddot{x} + 2\xi\omega_n\dot{x} + {\omega_n}^2 x = \frac{m}{M} e\omega^2 \sin(\omega t)$$

- $F_0 = \frac{F}{M} = \frac{me\omega^2}{M}$
- Centrifugal force

$$F_{centrifugal} = m\frac{v^2}{r}$$

- Amplitude of oscillations

$$X = \frac{me}{M}\frac{r^2}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

- Resonant frequency and amplitude

$r_p = \frac{\omega_r}{\omega_n} = \frac{1}{\sqrt{1-2\xi^2}}$ and $\frac{MX_r}{me} = \frac{1}{2\xi\sqrt{1-\xi^2}}$

Formula sheet

1. Equations of Motion

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F(t)\}$$

- Euler-Lagrange equations of a system having $q_1, q_2 \dots \dots . . q_n$ as degrees of freedom (*L*=*T*-*V*)

$$\frac{d(\frac{\partial L}{\partial \dot{q}_i})}{dt} - \frac{\partial L}{\partial q_i} = Q_{nci}$$

- Law of Cosines

$$a^2 = b^2 + c^2 - 2bc * \cos(A)$$

2. Eigenvalue Problem (Free vibrations)

$$\{x(t)\} = \{X\}e^{j\omega t}$$

- $Substitute\ EVP\ into\ EoM$
- $Solve\ for\ \omega_1\ and\ \omega_2\ (natural\ frequencies)$
- $Solve\ for\ X_1\ and\ X_2\ (always\ set\ X_1 = 1)\ to\ find\ Modal\ Vectors$

3. Orthogonality/Orthogonal transformation (Free vibrations)

- $Check\ for\ orthogonality$

$$[\phi] = \{Modal\ Vector_{\omega=\omega 1}, Modal\ Vector_{\omega=\omega 2}\}$$

$$[\phi]^T[M][\phi] = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$M_{11} = \{X\}_1{}^T[M]\{X\}_1$$

$$M_{12} = \{X\}_1{}^T[M]\{X\}_2 = 0$$

$$M_{21} = \{X\}_2{}^T[M]\{X\}_1 = 0$$

$$M_{22} = \{X\}_2{}^T[M]\{X\}_2$$

$$[\phi]^T[K][\phi] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$K_{11} = \{X\}_1{}^T[K]\{X\}_1$$

$$K_{12} = \{X\}_1{}^T[K]\{X\}_2 = 0$$

$$K_{21} = \{X\}_2{}^T[K]\{X\}_1 = 0$$

$$K_{22} = \{X\}_2{}^T[K]\{X\}_2$$

- $Orthogonal\ transformation$:

$$\{x(t)\} = [\Phi]\{y(t)\}$$

$$M_{11}\ddot{y}_1 + K_{11}y_1 = 0$$

$$M_{22}\ddot{y}_2 + K_{22}y_2 = 0$$

4. General Solution/Normal Mode Vibrations

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (A\cos\omega_1 t + B\sin\omega_1 t)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\omega=\omega 1} + (C\cos\omega_2 t + D\sin\omega_2 t)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\omega=\omega 2}$$

5. Absorber (forced vibrations)

$$[M]\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + [K]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A\cos\Omega t \\ B\cos\Omega t \end{pmatrix}$$

- In complex form:

$$[M]\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + [K]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Ae^{j\Omega t} \\ Be^{j\Omega t} \end{pmatrix}$$

- Steady-state solutions (X_1, X_2)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{j\Omega t}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

- Inverse of a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\mathrm{ad} - \mathrm{bc}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$