

Mathematics Qualifying Exam

Study Material

The candidate is expected to have a thorough understanding of engineering mathematics topics. These topics are listed below for clarification. Not all instructors cover exactly the same material during a course, thus it is important for the candidate to closely examine the subject areas listed below. The textbook listed below is a good source for the review and study of a majority of the listed topics. One final note, the example problems made available to the candidates are from past exams and do not cover all subject material. These problems are not to be used as the only source of study material. The topics listed below should be your guide for what you are responsible for knowing.

Suggested textbook:

Advanced Engineering Mathematics, E. Kreyszig, (John Wiley & Sons, pub)

Topic areas:

1. Partial Differential Equations
 - a. Separation of variables
 - b. Numerical solution using finite difference methods
2. Linear Algebra
 - a. Basic matrix algebra
 - b. Simultaneous linear equations
3. Ordinary Differential Equations
 - a. Solution of linear constant coefficient equations

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Problem 1.

Consider the vector space P_3 of quadratic polynomials in one real variable. Let $L = I - 2D + D^2 : P_3 \rightarrow P_3$, where D denotes differentiation with respect to the variable, be the linear transformation given by: $L(p) = p - 2p' + p''$, where $p'(x)$ and $p''(x)$ denote the first and second derivatives of $p(x)$ respectively. Here I denotes the identity map, $I(p) = p$.

- (a) Find the matrix A representing L with respect to the standard basis $E = \{1, x, x^2\}$. (Hint: A is the matrix that post-multiplies the row vector $[1 \ x \ x^2]$ to give $[1 \ x \ x^2]A = [L(1) \ L(x) \ L(x^2)]$.)
- (b) Find the transformation matrix from the basis $F = \{1 - x, 2x^2 - x, x^2 - 1\}$ to the basis E .
- (c) Find the matrix B representing the linear map L with respect to the basis F .

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Problem 2.

Solve the following problem:

$$\frac{\partial u}{\partial x} + 2x \frac{\partial u}{\partial t} = 2x, \quad u(x,0) = 1, \quad u(0,t) = 1$$

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Problem 3.

Find the work done by the force $\mathbf{F} = 2xy^3 \sin(z)\mathbf{i} + 3x^2y^2 \sin(z)\mathbf{j} + x^2y^3 \cos(z)\mathbf{k}$ in the displacement around the curve of intersection of the paraboloid $z = x^2 + y^2$ and the cylinder $(x-1)^2 + y^2 = 1$

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Problem 4.

Find the general solution of the following ODE

$$(x^2 - 1)y'' - 2xy' + 2y = 0$$

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Problem 1.

Find e^A and e^{At} and show by direct differentiation that $d[e^{At}]/dt = Ae^{At}$, given that

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Problem 2.

Let U be an $m \times m$ matrix, V be an $n \times n$ matrix, $m > n$, and let

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ O \end{bmatrix}$$

where Σ_1 is an $n \times n$ diagonal matrix with diagonal entries $\sigma_1, \sigma_2, \dots, \sigma_n$ and O is the $(m - n) \times n$ zero matrix. The columns of U are denoted \mathbf{u}_i , $i = 1, \dots, m$, and the columns of V are denoted \mathbf{v}_j , $j = 1, \dots, n$.

(a) If $U = [U_1 \ U_2]$, where U_1 has n columns, show that

$$U\Sigma = U_1\Sigma_1.$$

(b) Show that if $A = U\Sigma V^T$, then A can be expressed as an outer product expansion of the form

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \dots + \sigma_n \mathbf{u}_n \mathbf{v}_n^T.$$

(c) If U and V are orthogonal matrices, then obtain the diagonalization, and hence the eigenvalues of $A^T A$.

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Problem 3.

Evaluate

$$\int x^3 J_0(x) dx$$

($J_0(x)$ is Bessel function of the first kind of order zero)

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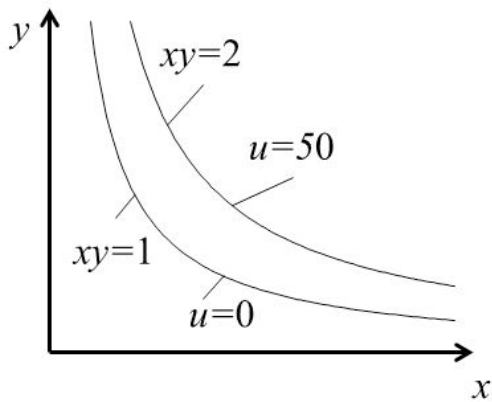
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Problem 4.

Solve Laplace equation $u_{xx} + u_{yy} = 0$ on the curved strip shown in the figure below



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Fall 2013

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Problem 1.

Find general solution of any three of the following four homogeneous differential equations

1. $x^2 y'' - 5xy' + 9(x^6 - 8)y = 0$
2. $y'' + 4x^{-1}y' + \frac{5}{2}x^{-2}y = 0$
3. $y' - 2xy \tan(x^2) = 0$
4. $(x^2 + x)'' + (4x + 2)y' + 2y = 0$

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Problem 2.

a) The three eigenvalues of a skew-symmetric 3×3 matrix A , (i.e., $A^T = -A$), are 0 and a pair of conjugate imaginary eigenvalues ($\pm j\phi$). Show that $I + A$ is invertible, where I denotes the 3×3 identity matrix.

b) Show that for the skew-symmetric matrix A , the matrix

$$Q = (I - A)(I + A)^{-1}$$

is orthogonal. This is also known as *Cayley's transform* relating skew-symmetric and orthogonal matrices. What are the eigenvalues of Q , given the eigenvalues of A in (a)?

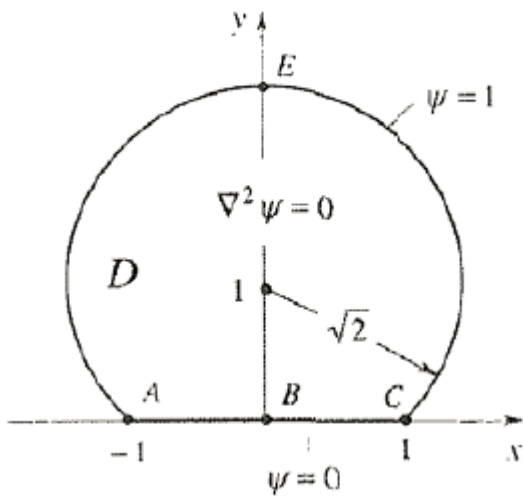
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Problem 3. Find the steady-state temperature distribution $\psi(x, y)$ in the cut-off disk shown below subject to the indicated boundary conditions.



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Problem 4.

Find the work done by the force $F = 2xy^3 \sin z \mathbf{i} + 3x^2y^2 \sin z \mathbf{j} + x^2y^3 \cos z \mathbf{k}$ in the displacement around the curve of intersection of the paraboloid $z = x^2 + y^2$ and the cylinder $(x - 1)^2 + y^2 = 1$

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Spring 2010

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This portion of the qualifying exam is closed book. YOU MAY NOT USE A PROGRAMMABLE CALCULATOR FOR THIS EXAM.

Work 3 of the 4 problems. Be very clear which 3 you want graded (see below). It is not acceptable to work all 4 problems and hope that the graders pick out the best worked three.

I want problems #____, #____, and #____ graded.

Be sure to put your name on all papers handed in, including this cover sheet.

1. Solve the conduction problem

$$\alpha^2 u_{xx} = u_t \quad 0 < x < L; \quad 0 < t < \infty$$

$$u_x(0, t) = -1; \quad u_x(L, t) = 0; \quad u(x, 0) = 0$$

HINT: To proceed successfully, change from $u(x, t)$ to $v(x, t)$ according to

$$u(x, t) = \frac{(x-L)^2}{2L} + v(x, t)$$

and split $v(x, t)$ by superposition into $v(x, t) = v_1(x, t) + v_2(x, t)$ so that

$$\alpha^2 v_{1xx} = v_{1t};$$

$$\alpha^2 v_{2xx} = v_{2t} - \frac{\alpha^2}{L}$$

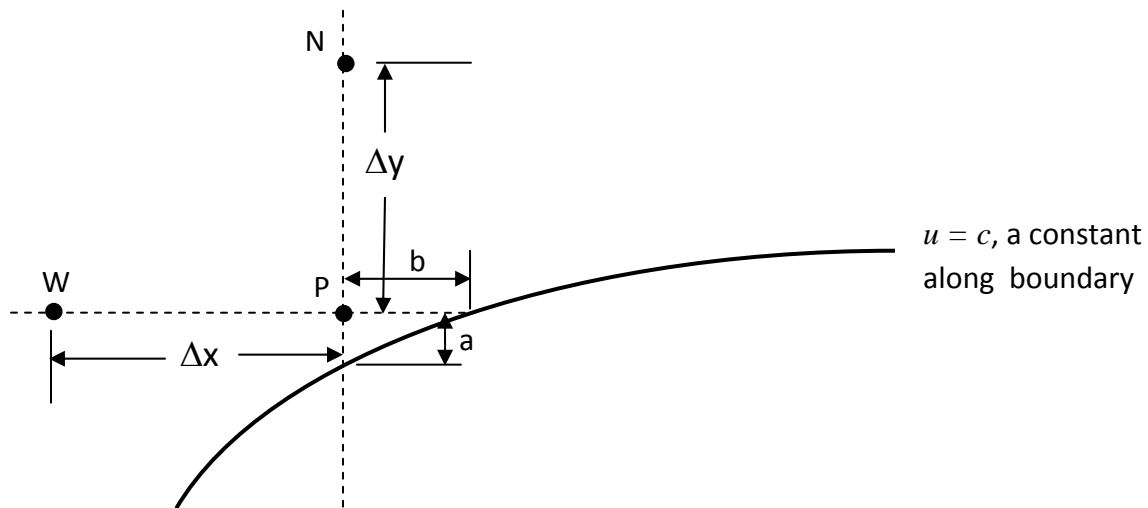
The solution for $v_2(x, t)$ can be found as a function of t alone

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2. Derive the finite difference approximation to Laplace's equation at node P in terms of the neighboring nodes and the boundary shown. Develop the nodal equation starting with the appropriate Taylor series expansions. What is the accuracy of your form?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



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3. Obtain the general solution of the following system of ODE's

$$\begin{aligned}x'' + y'' &= x \\ 3x'' - y'' &= y + 6\end{aligned}$$

4. From the eigenvectors of the given 4×4 matrix obtain an orthogonal basis for R^4 .

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix}$$

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I want problems #____, #____, and #____ graded.

Be sure to put your name on all papers handed in, including this cover sheet.

1. Does the system of linear equations

$$x_1 - x_2 = 1$$

$$2x_1 + x_2 = 2$$

$$x_1 + x_2 = 3$$

have an exact solution? If your answer to the question is yes, find the exact solution. If your answer is no, find an approximate solution which has the least square error. Give details of your solution procedure.

2. Can an Euler-Cauchy differential equation (essentially an ODE with variable coefficients) be reduced to an ODE with constant coefficients by means of a suitable substitution? Take the second-order Euler-Cauchy ODE below:

$$x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + by = 0 \quad (1)$$

(where a & b are constants) and clearly demonstrate that this equation can be reduced (or not reduced) to the following general ODE with constant coefficients:

$$\frac{d^2y}{dt^2} + C \frac{dy}{dt} + Dy = 0 \quad (2)$$

using the substitution:

$$x = e^t \quad (3)$$

If this works, express the values of the constants C and D in terms of the original constants: a & b .

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3. Solve the following 1-D diffusion problem. The boundary condition at $x = 0$ is $\frac{\partial u}{\partial x} = 0$, while the boundary condition at $x = L$ is $\frac{\partial u}{\partial x} = \beta$, where β is a positive constant. The initial condition is $u(x, 0) = 0$.

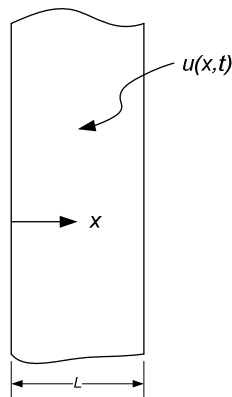
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

There is more than one way to solve this problem. If the hints below are useful, then go ahead and make use of them. If you have a method for solving this problem that does not make use of Hint #2, that is fine.

Hint #1: There is no steady state solution.

Hint #2: One solution method involves an assumed solution of the form:

$$u(x, t) = \psi(x, t) + \phi(t) + \theta(x)$$



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4. Numerically solve the first-order ODE

$$y'(x) = f(x, y) \text{ with initial condition } y(0) = y_0$$

one can use the following 4th-order Runge-Kutta method:

For the i^{th} step, $i = 0, 1, 2, \dots$, calculate the following intermediate variables

$$\begin{aligned}k_1 &= h f(x_i, y_i) \\k_2 &= h f(x_i + 0.5h, y_i + 0.5k_1) \\k_3 &= h f(x_i + 0.5h, y_i + 0.5k_2) \\k_4 &= h f(x_i + h, y_i + k_3)\end{aligned}$$

Then calculate the new y value (i.e., the approximation of the solution $y(x)$ at step $x_{i+1} = x_i + h$)

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where h is the increment of the independent variable x at each step.

Based on the above-described numerical procedure, write your procedure to solve the following 3rd-order ODE

$$y''' - 2y'' + y' + 3x^2 = 0 \text{ with initial conditions}$$

with initial conditions

$$\begin{aligned}y(0) &= y_0 \\y'(0) &= y'_0 \\y''(0) &= y''_0\end{aligned}$$

In your solution procedure, use a top bar to indicate an array or vector. For example, \bar{z} means that the z variable is an array or vector.

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Work 3 of the 4 problems. Be very clear which 3 you want graded (see below). It is not acceptable to work all 4 problems and hope that the graders pick out the best worked three.

I want problems #____, #____, and #____ graded.

Be sure to put your name on all papers handed in, including this cover sheet.

1. Solve the following ODE:

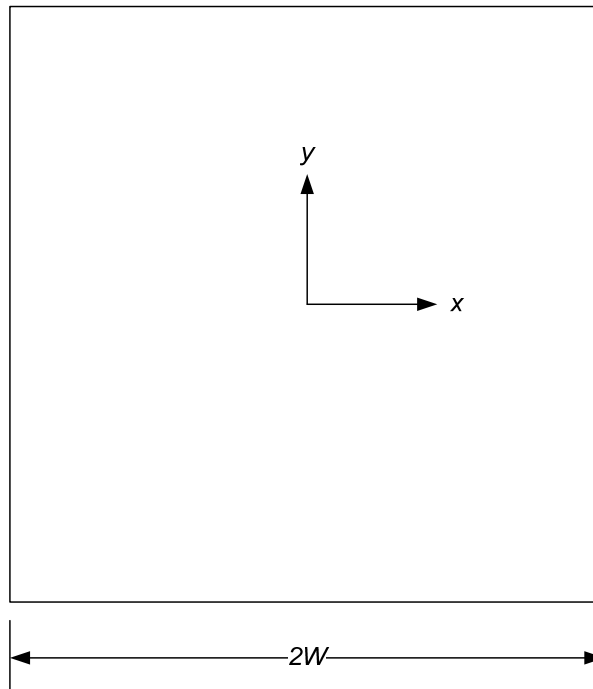
$$\frac{dy}{dx} = (2x + 3x^2y - y^2 \cos x)/(2y \sin x - x^3 + \ln y)$$

with the condition: $y(0) = e$

Note: $\int \ln y = y \ln y - y$

2. Solve Poisson's equation on the 2-D square region shown below. All boundary conditions are $u = 0$. \dot{q}_0 is a positive constant. You may use symmetry if it makes the problem easier for you.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\dot{q}_0$$



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3. For a given matrix

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Find a matrix B which can diagonalize matrix A through similarity transformation;

Determine the resulting diagonal matrix.

Let I be the identity matrix of the same dimension as A . No matter whether A is singular or not, $(A+cI)$ must be nonsingular as long as the constant c is larger than a number. What is the relation between this number and the eigenvalues of A ?

4. Write a
- self-contained
- algorithm or procedure which can
- numerically
- solve the following set of ordinary differential equations

$$\frac{d^2x}{dt^2} - 2\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \frac{d^2y}{dt^2} - \left(\frac{dy}{dt}\right) + x = 0$$

$$\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) - \frac{d^2y}{dt^2} + 2\left(\frac{dx}{dt}\right) - y = 0$$

with the following known initial conditions

$$x|_{t=0} = x_0, \quad \left.\frac{dx}{dt}\right|_{t=0} = \dot{x}_0, \quad y|_{t=0} = y_0, \quad \left.\frac{dy}{dt}\right|_{t=0} = \dot{y}_0$$

Note: as a numerical procedure, your solution should be presented as $x(t_k) = x(t_{k-1}) + \dots$

and $y(t_k) = y(t_{k-1}) + \dots$ where t_k ($k = 1, 2, 3, \dots$) is the k^{th} time step. Any numerical integration scheme may be used in your solution as long as it is correctly presented.

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This portion of the qualifying exam is **closed** book. You may have a simple 4-function (+, -, *, /) calculator – no programmable calculators.

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I want problems #____, #____, and #____ graded.

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1. Solve the following ordinary differential equation

$$\frac{d^3 y}{d^3 x} - 2 \frac{d^2 y}{d^2 x} + 4 \frac{dy}{dx} - 8 y = 8 e^{2x}$$

with:

$$y(0) = -1$$

$$\frac{dy}{dx}(0) = 31$$

$$\frac{d^2 y}{d^2 x}(0) = 32$$

2. Given:

$$\frac{\partial}{\partial x} \left(k(T) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k(T) \frac{\partial T}{\partial y} \right) = f(x, y); \quad 0 \leq x \leq 1; \quad 0 \leq y \leq 2$$

where

$$k(T) = k_o + k_1 T \quad k_o, k_1 \text{ constants}$$

and

$$T(0, y) = 0$$

$$T(x, 0) = 0$$

$$\frac{\partial T}{\partial x}(1, y) = 0$$

$$\frac{\partial T}{\partial y}(x, 2) = 0$$

Set up the appropriate equations for solving the above system for T using finite differences. Outline the solution procedure.

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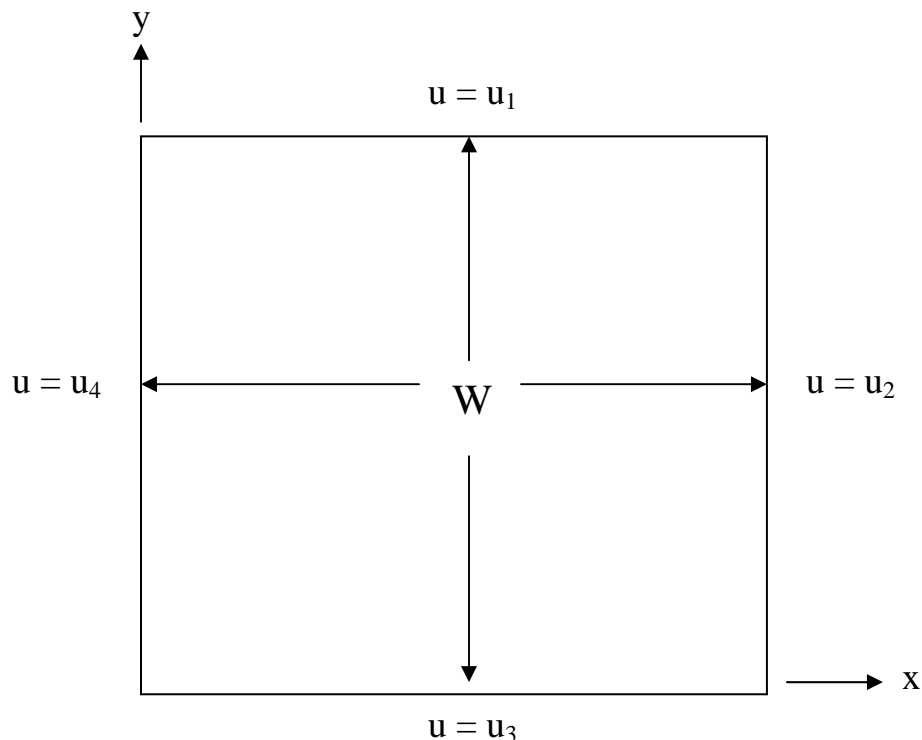
3. (Note: closed-book exam, NO CALCULATORS are allowed)

For a given matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

- Find a matrix \mathbf{B} which can diagonalize matrix \mathbf{A} through a similarity transformation.
 - Determine the resulting diagonal matrix.
 - Let \mathbf{I} be the identity matrix of the same dimension as \mathbf{A} . If \mathbf{A} is singular, then the matrix $(\mathbf{A} + c\mathbf{I})$ must be nonsingular as long as the constant c satisfies certain conditions. What are these conditions?
4. Solve Laplace's equation on the 2-D square region shown below. The boundary conditions are all of the type $u = c$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



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Work 3 of the 4 problems. Be very clear which 3 you want graded (see below). It is not acceptable to work all 4 problems and hope that the graders pick out the best worked three.

I want problems #____, #____, and #____ graded.

Be sure to put your name on all papers handed in, including this cover sheet.

1. The following PDE and boundary conditions describe the 1-D transient heat transfer for a fin of constant cross-section and length L . The fin is simultaneously being heated by a uniform heat flux and cooled by convection. All parameters $(\alpha, m^2, T_\infty, T_i, T_b, k, h, q'', P, A_c)$ are constant. Determine the temperature $T = T(x, t)$.

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - m^2 (T - T_\infty) + \frac{q'' P}{k A_c}$$

$$T(x, 0) = T_i \quad T(0, t) = T_b \quad -k \left. \frac{\partial T}{\partial x} \right|_{x=L} = h [T(L, t) - T_\infty]$$

2. Determine if the following ODE is exact or not:

$$[2 \exp(2x) \cos(y) + yx] dx + [x^2/2 - \exp(2x) \sin(y)] dy = 0$$

If it is exact, solve it for the case where $x(\pi) = 1$. If it is not exact, state how you would then try to solve it.

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3. (Note: closed-book exam, calculators with an eigenvalue solver are not allowed)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- Find the rank of the matrix.
- If one of the eigenvalues of the matrix is known to be -1.1168 , what are the other eigenvalues?
- Find a vector or vectors which span the null space of the matrix.

Hint: The problem may be solved without explicitly solving the characteristic equation. You have to show your solution procedure in detail.

4. Given:

$$\frac{\partial}{\partial x} \left(k(x) \frac{\partial T}{\partial x} \right) + k(x) \frac{\partial^2 T}{\partial y^2} = 0; \quad 0 \leq x \leq 1; \quad 0 \leq y \leq 2$$

where

$$k(x) = \begin{cases} k_0, & 0 \leq x \leq 0.5 \\ k_1, & 0.5 \leq x \leq 1 \end{cases} \quad k_0, k_1 \text{ constants}$$

and

$$T(0, y) = 25$$

$$T(x, 0) = 10$$

$$k(1) \frac{\partial T}{\partial x}(1, y) = h(200 - T(1, y)); \quad h \text{ constant}$$

$$\frac{\partial T}{\partial y}(x, 2) = 0$$

Set up the appropriate equations for solving the above system for T using finite differences. Outline the solution procedure.

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Work 3 of the 4 problems. Be very clear which 3 you want graded (see below). It is not acceptable to work all 4 problems and hope that the graders pick out the best worked three.

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Be sure to put your name on all papers handed in, including this cover sheet.

1. Solve the following ODE by the method of Laplace Transforms. Derivatives are with respect to time and $\delta(t)$ is the Delta or Impulse function.

$$y'' - y' + y = t \delta(t - 1) \quad \text{with } y(0) = y'(0) = 0$$

2. The following PDE and boundary conditions describe the 1-D transient heat transfer for a fin of constant cross-section and length, L . All parameters (α , m^2 , T_∞ , T_i , T_b , k , h) are constant. Determine the temperature, $T = T(x, t)$.

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - m^2 (T - T_\infty)$$

$$T(x, 0) = T_i \quad T(0, t) = T_b \quad -k \left. \frac{\partial T}{\partial x} \right|_{x=L} = h [T(L, t) - T_\infty]$$

Qualifying Exam: Mathematics

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3. Given:

$$\frac{\partial}{\partial x} \left(k(T) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k(T) \frac{\partial T}{\partial y} \right) = 0; \quad 0 \leq x \leq 1; \quad 0 \leq y \leq 2$$

where

$$k(T) = k_0 + k_1 T; \quad k_0, k_1 \text{ constants}$$

and

$$T(0, y) = 25$$

$$T(x, 0) = 10$$

$$k(T(1, y)) \frac{\partial T}{\partial x}(1, y) = h (200 - T(1, y)); \quad h \text{ constant}$$

$$\frac{\partial T}{\partial y}(x, 2) = 0$$

Set up the appropriate equations for solving the above system for T using finite differences. Outline the solution procedure.

4. Given the nonsingular matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

- a). (75%) Find a 2×2 transformation matrix \mathbf{X} which diagonalizes the \mathbf{A} matrix and also find the resulting diagonal matrix.
- b). (25%) If $\mathbf{B} = \mathbf{A}^7 = \mathbf{A}\mathbf{A}\mathbf{A}\mathbf{A}\mathbf{A}\mathbf{A}\mathbf{A}$, find a 2×2 transformation matrix \mathbf{Y} which diagonalizes the \mathbf{B} matrix? Hint: no need to repeat the procedure for solving problem a).

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Fall 2006

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I want problems # ____, # ____, and # ____ graded.

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1. Given: (do parts a and b)

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}; \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}; \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

- a. Evaluate whether $\mathbf{Ax}=\mathbf{b}_1$ has a unique solution, many solutions, or no solution. Solve for \mathbf{x} if $\mathbf{Ax}=\mathbf{b}_1$ has either a unique solution or many solutions.
- b. Evaluate whether $\mathbf{Ax}=\mathbf{b}_2$ has a unique solution, many solutions, or no solution. Solve for \mathbf{x} if $\mathbf{Ax}=\mathbf{b}_2$ has either a unique solution or many solutions.

Show all work.

Qualifying Exam: Mathematics

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2. Finite Differences (do parts a and b).

a. Given:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}; \quad 0 \leq t; \quad 0 \leq x \leq 5$$

with the following conditions on $T(x,t)$:

$$T(x,0) = f(x);$$

$$T(0,t) = 0;$$

$$\frac{\partial T}{\partial x}(5,t) = h(T_\infty - T(5,t))$$

Use finite differences to develop the matrix equations to approximate the solution of the above problem, for each time step, using an implicit method. Show all work.

b. Given:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = g(x,y); \quad 0 \leq x \leq 1; \quad 0 \leq y \leq 2; \quad 0 \leq z \leq 2$$

with the following conditions on $T(x,y,z)$:

$$T(0,y,z) = f(y,z); \quad T(1,y,z) = 0;$$

$$\frac{\partial T}{\partial y}(x,0,z) = 0; \quad \frac{\partial T}{\partial y}(x,2,z) = 0;$$

$$T(x,y,0) = 0; \quad T(x,y,2) = 0$$

Set up the finite difference equations to approximate the solution of the above problem. Explain how the resulting finite difference equations should be solved.

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3. The following equation describes linearized subsonic 2-D compressible flow.

$$\beta^2 \phi_{xx} + \phi_{yy} = 0$$

Here $\beta^2 = \sqrt{M_\infty^2 - 1}$ where M_∞ is the free stream Mach number, a constant. Apply separation of variables to solve for $\phi(x, y)$ given the following conditions:

$$\lim_{y \rightarrow \infty} \phi_x = 0$$

$$\phi_y(x, 0) = \frac{h}{l} 2\pi V_\infty \sin\left(\frac{2\pi x}{l}\right)$$

Here h, l and V_∞ are constants.

4. Solve the following ODE

$$\frac{dy}{dx} + 4x^2 y = (4x^2 - x)e^{-x^2/2}$$

with

$$y(0) = 1$$

Show all work.

Qualifying Exam: Mathematics**CLOSED BOOK**

This portion of the qualifying exam is **closed** book. You may have a calculator.

Work 3 of the 5 problems. Be very clear which 3 you want graded (see below). It is not acceptable to work all 4 problems and hope that the graders pick out the best worked three.

I want problems #____, #____, and #____ graded.

Be sure to put your name on all papers handed in, including this cover sheet.

1. Given:

$$y = a + b \sin x + c \sin 2x$$

with the data

| x , radians | y |
|---------------|------|
| 1 | 6.3 |
| 2 | -0.2 |
| 3 | 0.2 |
| 4 | 3.4 |

Set up the normal equations to find the values for a , b , and c that result in a least-squares fit of the equation to the data. Solve the resulting equations for a , b , and c . Show all work. You may use a calculator only for simple arithmetic calculations.

2. Given:

$$\mathbf{A} = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of \mathbf{A} . Show all work. You may use a calculator only for simple arithmetic calculations.

3. Solve the following ordinary differential equation:

$$\sin(y-x) dx + [\cos(y-x) - \sin(y-x)] dy = 0$$

with the condition: $y = 0$ when $x = \pi$

Qualifying Exam: Mathematics

CLOSED BOOK

4. A redundantly driven mechanical system can be modeled as an underdetermined linear system

$$\mathbf{Ax} = \mathbf{b}$$

where \mathbf{A} is an $m \times n$ matrix and $m < n$

Since such a system of linear equations can have infinitely many solutions of \mathbf{x} , an optimal solution can be obtained based on a specific application need. A commonly considered optimal solution is the one that \mathbf{x} has a minimum Euclidean norm. Show that the minimum-norm solution to the given linear system is

$$\mathbf{x} = \mathbf{A}^T (\mathbf{A}\mathbf{A}^T)^{-1} \mathbf{b}$$

where the super script T and -1 represent the transpose and inversion of the matrix, respectively. Also indicate the rank condition of \mathbf{A} under which such a solution exists.

Qualifying Exam: Mathematics

CLOSED BOOK

5. A rectangular metal bar is used to carry very large electrical currents. The bar has a cross-sectional dimensions $W \times H$ and is very long. The bar is initially at a uniform temperature of T_i . At time $t = 0$ a large current is allowed to flow through the bar resulting in a uniform volumetric energy generation

$$q \left[\frac{W}{m^3} \right]$$

The bottom of the bar rest on insulating material while the remaining three sides are subjected to natural convection cooling. Since the bar is very long all heat transfer may be treated as two dimensional. The first-law formulation for the problem yields the following PDE:

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{q}{k}$$

The thermal diffusivity α , and thermal conductivity k may be treated as constants. The boundary conditions are as follows (refer to figure):

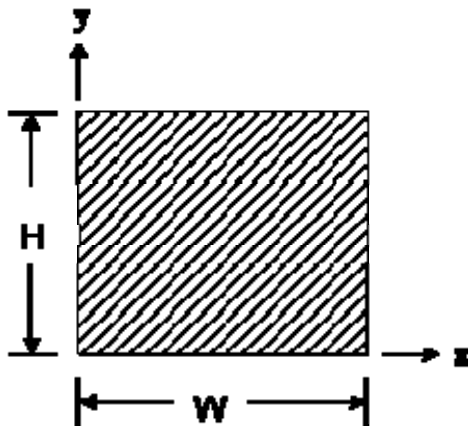
$$\frac{\partial T(x, 0, t)}{\partial y} = 0$$

$$\frac{\partial T(x, H, t)}{\partial y} = h [T(x, H, t) - T_\infty]$$

$$\frac{\partial T(0, y, t)}{\partial x} = h [T_\infty - T(0, y, t)]$$

$$\frac{\partial T(W, y, t)}{\partial x} = h [T(W, y, t) - T_\infty]$$

The heat transfer coefficient h , and ambient temperature T_∞ , may also be treated as constants.



Develop a solution by what ever means you choose. You may not have sufficient time to work out all the details, but be clear on your method.

NAME: _____

Spring 2005

Qualifying Examination
Subject: Mathematics

This portion of the qualifying exam is **closed** book. You are **not** allowed to use a calculator for the Mathematics Examination.

Work 3 of the 4 problems. Be very clear which 3 you want graded (see below). It is not acceptable to work all 4 problems and hope that the graders pick out the best worked three.

I want problems #____, #____, and #____ graded.

Be sure to put your name on all papers handed in, including this cover sheet.

Problem 1:

Solve the following partial differential equation by the method of Separation of Variables and obtain a general solution for a separation constant of zero.

$$\frac{\partial^3 U}{\partial x^3} - \left(\frac{2}{x^2}\right) \frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial y^2} = 0 \quad U = U(x,y)$$

Problem 2:

Given:

$$\mathbf{Ax} = \mathbf{b}; \quad \mathbf{A} = \begin{bmatrix} 6 & 10 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 2 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Find:

- a. The eigenvalues and eigenvectors of \mathbf{A} .
- b. \mathbf{A}^{-1}
- c. \mathbf{x}

Show all work. Perform all operations by hand (do not use a calculator).

Problem 3:

Determine which, if any, of the following differential equations are exact. Solve any that are exact, subject to the given boundary conditions:

$$4 dx + x^{-1} dy = 0 \quad \text{with } y(1) = -8$$

$$\cos \pi x \cos 2\pi y dx = 2 \sin \pi x \sin 2\pi y dy \quad \text{with } y(3/2) = 1/2$$

$$2 \sin \omega y dx + \omega \cos \omega y dy = 0 \quad \text{with } y(0) = \pi/2\omega$$

NAME: _____

Spring 2005

Problem 4:

Given:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k(T) \frac{\partial T}{\partial x} \right); \quad 0 \leq x \leq 1; \quad 0 < t$$

with

$$k(T) = 5 + 0.05T$$

and

$$T(x,0) = 25$$

$$T(0,t) = 25$$

$$\frac{\partial T}{\partial x}(1,t) = 2(200 - T(1,t))$$

Set up the appropriate equations for solving the above system using finite differences.
Outline the solution procedure.

NAME: _____

Fall 2004

Qualifying Examination
Subject: Mathematics

This portion of the qualifying exam is **closed** book. You are **not** allowed to use a calculator for the Mathematics Examination. Work 3 of the 4 problems. Be very clear which 3 you want graded (see below). It is not acceptable to work all 4 problems and hope that the graders pick out the best worked three.

I want problems #____, #____, and #____ graded.

Be sure to put your name on all papers handed in, including this cover sheet.

Problem 1:

Assume the following system of linear equations has more equations than unknowns

$$\mathbf{Ax} = \mathbf{b} \quad \text{where } \mathbf{A} \text{ is an } m \times n \text{ matrix and } m > n$$

Such a system may have a unique solution, many solutions or no solutions depending on matrix \mathbf{A} and vector \mathbf{b} .

- (1) Find the least-square approximation of \mathbf{x} from the system.
- (2) State the condition on \mathbf{A} under which the least-square approximation exists?
- (3) Under what condition does the system have a unique solution?

No proofs are needed for your answer to (2) and (3).

Problem 2:

Consider the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} = c^2 \frac{\partial^2 y(x,t)}{\partial x^2}$$

in which c is a constant. Put this equation in a finite difference form with all terms of second order accuracy so that the resulting finite difference equations can be solved implicitly. Do not attempt to solve the resulting finite difference equations.

Problem 3:

Find the general analytical solution of the following ordinary differential equation:

$$(x - e^\pi)^2 y'' + 7(x - e^\pi) y' - 7y = 0$$

then find the specific solution with the boundary conditions:

$$y(2e^\pi) = e^\pi \quad \text{and} \quad y(0) = -e^\pi$$

Problem 4:

Find the exact analytical solution for the following partial differential equation:

$$t \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = xt$$

with the following conditions:

$$u(x,0) = u(0,t) = 0$$

NAME: _____

Fall 2003

Qualifying Examination
Subject: Mathematics

This portion of the qualifying exam is **closed** book. You may have a calculator.

Work 3 of the 4 problems. Be very clear which 3 you want graded (see below). It is not acceptable to work all 4 problems and hope that the graders pick out the best worked three.

I want problems #____, #____, and #____ graded.

Be sure to put your name on all papers handed in, including this cover sheet.

Problem 1:

Solve the following partial differential equation set by the method of Laplace Transforms:

$$\frac{\partial u}{\partial x} + 2x \frac{\partial u}{\partial t} = 2x$$

| |
|--------------|
| $u(x,0) = 1$ |
| $u(0,t) = 1$ |

Problem 2:

Expand the function:

$$f(x) = \cos^2 x$$

in a half-range Fourier Cosine Expansion on the interval 0 to π .

Problem 3:

Find the eigenvalues and eigenvectors for the matrix **A** below. Express the eigenvectors so that they have a magnitude of one. Show all your work.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 4:

Consider the parabolic equation $\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2}$ subject to the following boundary condition

At $x = 0$: $u(0, y) + \frac{\partial u(0, y)}{\partial x} = 1$. Develop a nodal equation at $x = 0$ for the finite difference formulation of this PDE. You may choose either the explicit or implicit formulation. Your resulting nodal equation should be of order (Δx) accuracy. Please use u_0^j for the value of u at $x = 0$ and the j th y value. u_1^j is the value of u at $x = \Delta x$.

Qualifying Exam Spring 2003

Mathematics

This portion of the qualifying exam is **closed** book. You may have a calculator.

Work **all** three problems.

Problem 1:

Find the general solution of the following second order ordinary differential equation:

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = \sin^2 x$$

Problem 2:

Solve the following PDE which corresponds to a rod that is insulated along the lateral surface, whose left end is insulated, and its right end is exposed to convective heat transfer.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t} \quad 0 < x < a, \quad 0 < t$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad 0 < t$$

$$-\kappa \frac{\partial u}{\partial x}(a, t) = h[u(a, t) - T_1] \quad 0 < t$$

$$u(x, 0) = T_0 \quad 0 < x < a$$

Problem 3:

Given the function $f(t) = t^3$, $-1 \leq t \leq +1$

- a) Sketch the function over two periods.
- b) Find the Fourier Series representation of $f(t)$.
- c) What is the value of $f(t)$ given by the series at $t = \pm 1$? What is the mean value of the function over the period?

Qualifying Exam Fall 2002

Mathematics

This portion of the qualifying exam is closed book. Work 4 of the 5 problems.

Mathematics Qualifying Examination: Problem #1

Solve the following initial value problem:

$$\dot{x} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix} x \quad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Mathematics Qualifying Examination: Problem #2

Find the work done by the force \vec{F} in the displacement around the curve of the intersection of the paraboloid $z = x^2 + y^2$ and the cylinder $(x-1)^2 + y^2 = 1$.

$$\begin{aligned} \vec{F} = & \left[3x^2y^7 \sec(xyz) \tan(z) + x^3y^8z \sec(xyz) \tan(z) \tan(xyz) \right] \hat{i} \\ & + \left[7x^3y^6 \sec(xyz) \tan(z) + x^4y^7z \sec(xyz) \tan(z) \tan(xyz) \right] \hat{j} \\ & + \left[x^3y^7 \sec^2(z) \sec(xyz) + x^4y^8 \sec(xyz) \tan(z) \tan(xyz) \right] \hat{k} \end{aligned}$$

Mathematics Qualifying Examination: Problem #3

The Lorenz equations for fluid convection are given by:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz \quad \text{where } \sigma = 10, \quad r = 28 \quad \text{and } b = \frac{8}{3}$$

Describe as many numerical techniques as you can for solving these equations. Please comment on differences between methods. For an initial value of $(x, y, z) = (10, 10, 10)$, calculate the values of (x, y, z) for the first 2 time steps using a time increment of 0.1 sec.

Mathematics Qualifying Examination: Problem #4

Find the general solution for the 4-dimensional axisymmetric transient heat conduction equation by separation of variables.

$$\frac{\partial^2 T}{\partial r^2} + \frac{3}{r} \frac{\partial T}{\partial r} = \frac{1}{\alpha} \frac{\partial T}{\partial \tau} \quad T = f(r, \tau)$$

Mathematics Qualifying Examination: Problem #5

Find the solution to the following ordinary differential equation.

$$y^{IV} - 5y'' + 4y = 10\cos(x) \quad \begin{aligned} y(0) &= 2 \\ y'(0) &= 0 \\ y''(0) &= 0 \\ y'''(0) &= 0 \end{aligned}$$

Qualifying Exam Spring 1999

Mathematics

This portion of the qualifying exam is closed book. Work 3 of the 4 problems.

1. Given:

$$y = a + bx + cx^2$$

with the data

| x | y |
|-----|------|
| 1 | 6.1 |
| 2 | 14.9 |
| 3 | 28.1 |
| 4 | 45.0 |

Set up the normal equations to find the values for a , b , and c that result in a least-squares fit of the quadratic to the data. Solve the resulting equations for a , b , and c . Show all work.

2. Given:

$$\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

with the following conditions on $\Phi(x, y, t)$:

$$\begin{aligned}\Phi(x, y, 0) &= 100, \\ \Phi(0, y, t) &= 0, \quad \Phi(1, y, t) = 100, \\ \frac{\partial \Phi}{\partial y}(x, 0, t) &= 0, \quad \frac{\partial \Phi}{\partial y}(x, 2, t) = (\Phi - 50)\end{aligned}$$

Set up the finite difference equations to approximate the solution to the above problem. Explain how the resulting finite difference equations should be solved.

3. Using the equations and boundary conditions of Problem 2, perform a separation of variables on the partial differential equation and determine the Sturm-Liouville problem in the y -direction. Solve the resulting ordinary differential equation in y and determine the eigenfunctions and the eigenvalue equation for this problem. Explain how you would determine the eigenvalues.

4. Determine the general solution to the following ordinary differential equation.

$$xy'(x) + 2y(x) = 9x$$

Qualifying Exam Fall 1999

Mathematics

This portion of the qualifying exam is closed book. Work 3 of the 4 problems.

1. Given:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}; \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}; \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- Evaluate whether $\mathbf{Ax} = \mathbf{b}_1$ has a unique solution, any solution, or no solution. Solve for \mathbf{x} if $\mathbf{Ax} = \mathbf{b}_1$ has either a unique solution or any solution.
- Evaluate whether $\mathbf{Ax} = \mathbf{b}_2$ has a unique solution, any solution, or no solution. Solve for \mathbf{x} if $\mathbf{Ax} = \mathbf{b}_2$ has either a unique solution or any solution.

Show all work.

2. Given:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

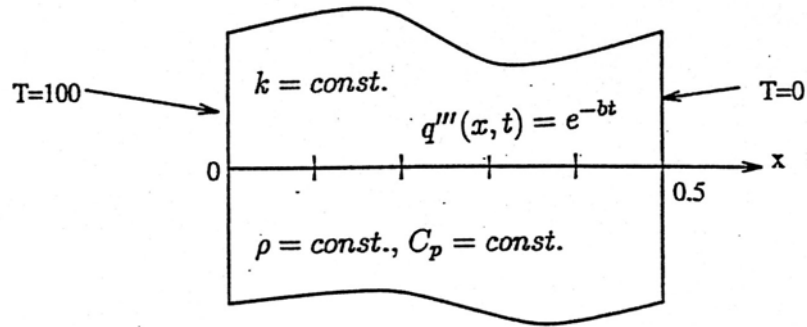
Find the eigenvalues and eigenvectors of \mathbf{A} . Show all work.

3. find the general solution to

$$\frac{dy(x)}{dx} + 4y(x) = \cos x$$

using the integrating factor technique. Check your results. Show all your work.

4. Consider the heat transfer in a slab, as shown.



If the initial condition is $T(x, 0) = 50^\circ\text{C}$ and the boundary conditions are $T(0, t) = 100^\circ\text{C}$ and $T(0.5, t) = 0^\circ\text{C}$, determine the general solution using transform techniques to the energy equation

$$\rho C_p \frac{\partial T(x, t)}{\partial t} = k \frac{\partial^2 T(x, t)}{\partial x^2} + q'''(x, t)$$

in which $\rho = \text{const.}$, $C_p = \text{const.}$, $k = \text{const.}$ and the heat generation term $q'''(x, t) = e^{-bt}$, where $b = \text{const.}$ Note that all temperatures are in $^\circ\text{C}$.

Qualifying Exam Spring 1998

Mathematics

This portion of the qualifying exam is closed book. Work all problems.

Problem No.1 Solve the following ordinary differential equation using Laplace transforms if the initial conditions are $y(0) = 0$ and $\dot{y}(0) = 0$:

$$\ddot{y}(t) + y(t) = 3\cos 2t$$

Problem No.2

Consider the time dependent heat transfer in a slab. The heat transfer equation is

$$\rho C_p \frac{\partial T(x, t)}{\partial t} = k \frac{\partial^2 T(x, t)}{\partial x^2}$$

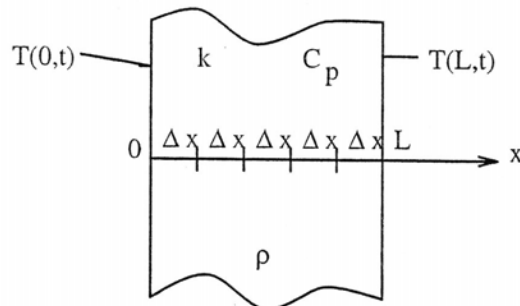
in which ρ is the mass density of the slab material, C_p is the specific heat of the material and k is the thermal conductivity coefficient of the material. Consider these constant and let

$$\alpha = \frac{k}{\rho C_p}$$

so that the heat transfer equation becomes

$$\frac{\partial T(x, t)}{\partial t} = \alpha \frac{\partial^2 T(x, t)}{\partial x^2}$$

Put this equation in finite difference form by assuming a central difference approximation to the second derivative in x term and a backward difference approximation to the first derivative in time term. Assume that the values of Δx are uniform (i.e constant) as are the values of Δt . Assuming the boundary conditions and the initial condition are known and using 4 internal nodes, show that such a formulation yields a set of linear algebraic equations in the unknown nodal point temperatures at each time step.



Qualifying Exam Fall 1998

Mathematics

This portion of the qualifying exam is closed book. Work all problems.

1. Given:

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -18 \\ 10 \end{bmatrix}$$

Find:

- The solution vector x using Gauss Elimination
- The eigenvalues of the matrix
- The eigenvectors of the matrix

Show all work.

2. Given:

$$\frac{\partial^2 \Phi}{\partial t^2} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

with the following conditions on $\Phi(x,y,t)$:

$$\begin{array}{lll} \Phi(x, y, 0) = 1 & \Phi(0, y, t) = 0 & \frac{\partial \Phi}{\partial x}(x, 0, t) = 0 \\ \frac{\partial \Phi}{\partial t}(x, y, 0) = 0 & \Phi(a, y, t) = 0 & \frac{\partial \Phi}{\partial y}(x, b, t) = 0 \end{array}$$

Find $\Phi(x,y,t)$ using separation of variables. Show all work.

Qualifying Exam Spring 1997

Mathematics

This portion of the qualifying exam is closed book. Work all problems.

1.) Solve the following partial differential equation

$$\frac{\partial w(x, t)}{\partial t} = \frac{\partial^2 w(x, t)}{\partial x^2}$$

having boundary conditions

$$w(0, t) = 0$$

$$\frac{\partial w(b, t)}{\partial x} = 0$$

and an initial condition

$$w(x, 0) = 100$$

Assume b is a constant.

3. Given the following equation and data:

$$y = ax + b$$

| x | y |
|-----|-----|
| 0.0 | 1.0 |
| 1.0 | 3.2 |
| 2.0 | 4.6 |
| 3.0 | 7.3 |

Find the coefficients a and b which result in the least squares fit of the equation to the data by the following steps:

- Formulate your problem in terms of a non-square matrix equation.
- Derive the corresponding normal equation.
- Solve the resulting normal equation for a and b .

Qualifying Exam Spring 1996

Mathematics

This portion of the qualifying exam is closed book. Work all problems.

1. Given:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

with the following initial and boundary conditions on $T(x,y)$:

$$\frac{\partial T}{\partial x}(0,y) = 0$$

$$\frac{\partial T}{\partial x}(a,y) = 0$$

$$T(x,0) = 0$$

$$T(x,b) = f(x)$$

Find the analytical solution for $T(x,y)$ using separation of variables.

2. Given the system of equations:

$$\mathbf{Ax} = \mathbf{b}$$

where \mathbf{A} is a n by m matrix, \mathbf{x} is a m by 1 and \mathbf{b} is n by 1.

- What are the constraints on \mathbf{A} for a unique solution for \mathbf{x} to exist?
- Show the least squares solution for the case of $n > m$ with rank = m .
- What does the least squares solution minimize?

3. The differential equation for the deflection of a critically damped spring-mass system is given by

$$m \frac{d^2 y}{dt^2} + c \frac{dy}{dt} + ky = 0$$

where $c^2 = 4mk$. Consider the following initial conditions on $y(t)$:

$$y(0) = 0$$

$$\frac{dy}{dt}(0) = 1$$

Find the analytical solution for $y(t)$. Sketch a typical solution for these initial conditions.

Qualifying Exam Spring 1995

Mathematics

This portion of the qualifying exam is closed book. Work all problems.

1. Given:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

with the following initial and boundary conditions on $T(x,t)$:

$$T(x,0) = 0$$

$$\frac{\partial T}{\partial x}(0,t) = 0$$

$$T(L,t) = f(t)$$

Find the analytical solution for $T(x,t)$ using separation of variables.

2. Solve

$$\begin{bmatrix} 1 & 4 & 2 \\ -2 & -8 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 32 \\ 1 \end{bmatrix}$$

for u, v, w .

3. The differential equation for the deflection of an underdamped (i.e., $c^2 < 4mk$) spring-mass system is given by

$$m \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} + ky = 0$$

with the following initial conditions on $y(t)$:

$$y(0) = 1$$

$$\frac{\partial y}{\partial t}(0) = 0$$

Find the analytical solution for $y(t)$.

Qualifying Exam Spring 1993

Mathematics

This portion of the qualifying exam is closed book. Work 4 of the 5 problems.

Analysis Question #1

Problem 1a. Find the vector $\bar{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$ of minimum length

$$l = (y_1^2 + y_2^2 + y_3^2)^{1/2}$$

that satisfies the equations

$$y_1 - y_2 = 1$$

and

$$y_2 - y_3 = -1.$$

Problem 1b. Consider the following system of equations:

$$2x_1 - x_2 = 1$$

$$2x_1 + 2x_2 = 1$$

$$-x_1 + 2x_2 = 1$$

There are no values for x_1, x_2 for which all three equations are satisfied exactly. Find the values for x_1, x_2 that minimize the "error" E defined as

$$E = (2x_1 - x_2 - 1)^2 + (2x_1 + 2x_2 - 1)^2 + (-x_1 + 2x_2 - 1)^2$$

Analysis Question #2

Problem 2. Using orthogonality (both problems are analogous to Fourier Series):

- a) determine $x_1, x_2, x_3 \in \mathbb{R}$ so that

$$x_1 \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- b) determine the $C_n, n = 1, 2, \dots, \infty$ so that

$$\sum_{n=1}^{\infty} C_n \psi_n(x) = f(x) \quad \text{for } x \in [a, b]$$

if $f: [a, b] \rightarrow \mathbb{R}$ is continuous and the set $\{\psi_n\}_{n=1}^{\infty}$ is orthogonal with respect to the weight $W(x): [a, b] \rightarrow \mathbb{R}$ (you may assume the set $\{\psi_n\}_{n=1}^{\infty}$ is complete)

Analysis Question #3

Problem 3. Solve the following partial differential equation modeling heat conduction in a bar with insulated ends at $x=0$ and $x=l$ and initial temperature $T_0 x(l-x)$: *Use analytical methods.*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad \frac{\partial u}{\partial x}(l, t) = 0 \quad u(x, 0) = T_0 x(l-x)$$

Analysis Question #4

Problem 4. Show how to solve the following partial differential equation modeling heat conduction in a plate using numerical methods.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \sin\left(\frac{\pi x}{a}\right)$$

$$T(0,y) = 0 \quad T(a,y) = 0 \quad \frac{\partial T}{\partial y}(x,0) = 0$$

$$\frac{\partial T}{\partial y}(x,b) = T_\infty - T(x,b), \quad (T_\infty \text{ a constant})$$

Set up the appropriate equations for implementation on the computer and explain how to solve them.

Analysis Question #5

The differential equation modeling the deflection of a mass in a damped spring mass system is given by

$$m \frac{d^2 y}{dt^2} + c \frac{dy}{dt} + ky = 0$$

$$y(0) = y_0 \quad \frac{dy}{dt}(0) = 0$$

Solve for y using analytical techniques assuming the system is underdamped ($c^2 < 4mk$).

Qualifying Exam Spring 1991

Mathematics

This portion of the qualifying exam is closed book. Work all problems.

Question 1.

Suggested Solution Time: 60 minutes, 60 points

Solve the boundary value problem given below. Show all details of your solution. If you have time check your answer, or at least indicate what you can do to verify your answer.

$$\begin{aligned}\nabla^2 U &= 0 \text{ in } 0 < x < a, 0 < y < a \\ U(0, y) &= 0, 0 < y < a \\ U(a, y) &= 0, 0 < y < a \\ U(x, 0) &= 1 - 2|x - a/2|, 0 < x < a \\ U(x, a) &= 1 - 2|x - a/2|, 0 < x < a\end{aligned}$$

Question 2.

Suggested Solution Time: 25 minutes, 25 points

Find the cube root of 15 to five significant digits. Use any standard root finding algorithm. For an initial guess, use $x = 2.00000$. Your algorithm should only employ these arithmetic operations: multiplication, division, addition, and subtraction. Show the equation that you derive for $f(x)$. Indicate all possible answers.

Question 3.

Suggested Solution Time: 25 minutes, 25 points

Solve this system of equations by hand. You can use a calculator to perform simple arithmetic, but you should illustrate the logic for the solution method. DO NOT SOLVE THE SYSTEM OF EQUATIONS USING A PROGRAM. Once you have obtained a solution for the three constants u , v , and w substitute your solution back into the left hand side, and compute the right hand side vector. Compare the computed right hand side vector with the original right hand side. Explain any discrepancies. Compute the determinant of the coefficient matrix.

$$\begin{aligned}2u + 6v - 2 &= 8 \\ 8u + 6v + 8w &= 6 \\ 6u + 4v + 2w &= 6\end{aligned}$$

Question 4.

Suggested Solution Time: 10 minutes, 10 points

Match the equation in the left column with the appropriate descriptor from the right column. Place the letter of the descriptor beside each equation. *Scoring: 2.5 pts. for each correct answer, 0 pts. for no answer, and -2.5 pts. for each incorrect answer.*

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial t}$$

$$w \frac{dw}{dx} + w = e^{ax}$$

$$\frac{\tan(\mu)}{\mu} = 10$$

- (A) Linear Ordinary Differential Equation
- (B) Nonlinear Ordinary Differential Equation
- (C) Linear Algebraic Equation
- (D) Nonlinear Algebraic Equation
- (E) Transcendental Equation
- (F) Nonlinear Partial Differential Equation
- (G) Elliptic Partial Differential Equation
- (H) Parabolic Partial Differential Equation
- (I) Hyperbolic Partial Differential Equation

3. Solve the following ordinary differential equation

$$\ddot{y}(t) + y(t) = 3 \cos 2t$$

using Laplace transforms.

The initial conditions are: $y(0) = 0$

$$\dot{y}(0) = 0$$