Qualifying Exam, Fall 2021
Mathematics

* This is a closed-book test (with cheat sheets provided), and no calculator is allowed.

* Work THREE out of the four problems, and clarify which three you want graded.

I want problems #______, #______, and # ______ to be graded.

1. Solve the following initial value problem using the method of Laplace Transforms:

$$\frac{dy}{dt} - y = te^t \sin t, \quad y(0) = 0$$

2. Solve the following non-homogeneous boundary value problem for $u(x,t)$:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + e^{-x}, \quad 0 < x < 1, \quad t > 0,$$

where c is a constant, and with boundary and initial conditions given by

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = f(x).$$

Recall that the solution $v(x,t)$ of the corresponding homogeneous boundary value problem (i.e. the PDE without the e^{-x} term and with boundary and initial conditions $v(0,t) = v(1,t) = 0, v(x,0) = g(x)$) is given by the infinite sum $v(x,t) = \sum_{n=1}^{\infty} A_n e^{-c^2 n^2 \pi^2 t} \sin(n\pi x)$, where the A_ns are the Fourier coefficients of $g(x)$ given by

$$A_n = 2 \int_0^1 g(x) \sin(n\pi x)\, dx$$

(*Hint:* Do not waste time doing unnecessary things in this problem. You are not required to derive the infinite sum solution given above for $v(x,t)$. You just need to use it to solve the problem.)

3. Prove the following two results for a vector field $\vec{V}$ defined on a bounded domain D in $\mathbb{R}^3$, where ∂D denotes the boundary of D and $\hat{n}$ is the unit normal field (assume outward pointing) on ∂D.

   (a) If $\vec{\nabla} \cdot \vec{V} = 0$, show that

   $$\int\int\int_D \vec{V}\, d\mathcal{V} = \int\int_{\partial D} \vec{r}\left(\vec{V} \cdot \hat{n}\right)\, dS$$

   where $\vec{r} \equiv (x, y, z)$ is position vector.

   (b) Show that

   $$\int\int\int_D \vec{\nabla} \times \vec{V}\, d\mathcal{V} = \int\int_{\partial D} \hat{n} \times \vec{V}\, dS$$

   (*Hint:* For both problems, start with the right hand side, take components in each of the coordinate directions and then apply vector calculus results.)

4. Answer the following two questions on functions of a complex variable:

   (a) Determine, using the definition of an analytic function, if the following function is analytic at any point in the complex plane. If yes, then obtain the points at which it is analytic.

   $$f(z) = 2x^2 + y + i(y^2 - x)$$

   (b) Determine $\oint_C f(z)dz$, where

   $$f(z) = \frac{\cos z}{(z^2 + 16)(z^2 - 4)},$$

   and C is the unit circle $\mid z \mid= 1$.

FORMULAS AND EQUATIONS

- **Laplace Transform Properties and Formulas.**

$$\mathcal{L}\left(af(t)+bg(t)\right) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t)),$$

$$\mathcal{L}\left(e^{at}f(t)\right) = F(s-a),$$

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s},$$

$$\mathcal{L}\left(e^{at}t^n\right) = \frac{n!}{(s-a)^{n+1}}, \quad n = 0,1,2,\cdots,$$

$$\mathcal{L}\left\{f(t-b)u(t-b)\right\} = e^{-bs}F(s),$$

$$\mathcal{L}\left\{\delta(t-b)\right\} = e^{-bs},$$

$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^nF}{ds^n},$$

$$\mathcal{L}\left\{f*g\right\} = \mathcal{L}(f)\mathcal{L}(g), \quad (f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

- **Cauchy-Riemann equations.**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- **Fourier Series.**

For a periodic function on the interval $a \le x < b$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{2n\pi x}{b-a}\right),$$

$$a_0 = \frac{1}{b-a}\int_a^b f(x)dx,$$
$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx,$$
$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2n\pi x}{b-a}\right)dx$$

- **Vector Integral Theorems and Identities.**

  The divergence theorem of Gauss.

$$\int\int\int_D \vec{\nabla}\cdot\vec{V}\; d\mathcal{V} = \int\int_{\partial D} \vec{V}\cdot\hat{n}\; dS$$

$$\vec{\nabla}\cdot(f\vec{V}) = \vec{V}\cdot\nabla f + f\vec{\nabla}\cdot\vec{V},$$
$$\vec{\nabla}\cdot(\vec{W}\times\vec{V}) = \vec{V}\cdot(\nabla\times\vec{W}) - \vec{W}\cdot(\nabla\times\vec{V}),$$

  In the above, f is a function and $\vec{V},\vec{W}$ are vector fields.

  Scalar Triple Product Rule for Vectors.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})$$

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$