Qualifying Exam, Fall 2020
Mathematics

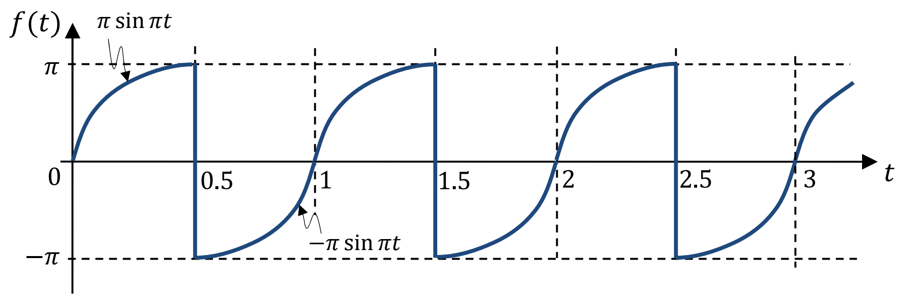
* This is a closed-book test (with a cheat sheet provided), and no calculator is allowed.
* Work THREE out of the four problems, and clarify which three you want graded.

I want problems #____________, #___________, and #____________ to be graded.

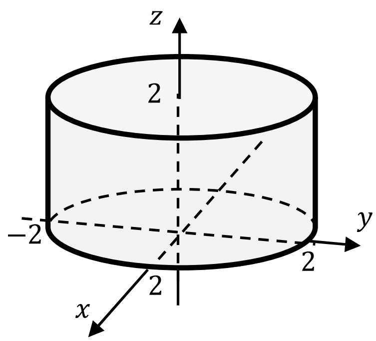
Problem 1. Write down the Fourier series of $f(t)$ depicted below up to the first three leading harmonic terms, and find the particular solution of the following 2nd-order ordinary differential equation due to resonance

$$x'' + x' + 16\pi^2 x = f(t)$$

where the prime denotes differentiation with respect to t.



Problem 2. Utilizing the divergence theorem of Gauss, evaluate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dA$ where $\boldsymbol{F} = 4xy^2\, \boldsymbol{i} + 4x^2y\boldsymbol{j} + z^2\boldsymbol{k}$ and the surface *S* consists of a cylindrical surface and the two disks of radius 2 at $z = 0$ and $z = 2$.



(Note: $dxdydz = rdrd\theta dz$)

Problem 3. The method of variation of parameters is utilized to find the particular solution of a 2nd-order linear ordinary differential equation $y'' + p(x)y' + q(x)y = r(x)$ by writing down the particular solution $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ where $y_1(x)$ and $y_2(x)$ are the two linearly independent homogeneous solutions.

(1) When $y'_p(x) = u(x)y'_1(x) + v(x)y'_2(x)$, show that the variable coefficients $u(x)$ and $v(x)$ can be obtained as $u(x) = -\int \frac{y_2(x)\, r(x)}{W(y_1,y_2)} dx$ and $v(x) = \int \frac{y_1(x)\, r(x)}{W(y_1,y_2)} dx$ where $W(y_1, y_2)$ is the Wronskian of $y_1(x)$ and $y_2(x)$.

(2) Then, find the solution of an Euler-Cauchy equation $x^2y'' - xy' - 3y = 4x, y(-1) = -2, y'(-1) = 0$ whose homogeneous solution can be expressed as $y_h(x) = c_1x^{-1} + c_2x^3$ where c_1 and c_2 are arbitrary constants that can be determined by the initial conditions.

Problem 4. By utilizing Laplace transforms and letting $W(x,s) = L[w(x,t)]$, solve the following 1D wave equation, $w_{tt} = w_{xx}$, with $w(x,0) = 2\sin \pi x$ and $w_t(x,0) = 0$, $w(0,t) = \sin \pi t$ and $\lim_{x\to\infty} w(x,t) < \infty$. Show that the obtained solution can be expressed in the form of $w(x,t) = \phi(x+t) + \psi(x-t)$, that is, as D'Alembert's wave solution.

Cheat Sheet

1. Definition: Laplace transform (LT) of $f(t)$ is defined as $\mathcal{L}[f(t)] \triangleq F(s) = \int_0^\infty e^{-st} f(t)dt$ and the inverse LT is $\mathcal{L}^{-1}[F(s)] = f(t)$.

2. Linearity: $\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$

3. Table for LT Pairs of Some Basic Functions

$f(t)$	$F(s)$	$f(t)$	$F(s)$
Unit impulse $\delta(t)$	1	Unit step $u(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$	t^n, $n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$

4. Convolution Integral: $\begin{cases} f_1(t) * f_2(t) \triangleq \int_0^t f_1(t-s) f_2(s) ds \equiv \int_0^t f_1(s) f_2(t-s) ds \triangleq f_2(t) * f_1(t) \\ \text{or} \quad \mathcal{L}[f_1(t) * f_2(t)] = \mathcal{L}[f_1(t)] \cdot \mathcal{L}[f_2(t)] = F_1(s) \cdot F_2(s) \end{cases}$

5. Shift in s- and t-Domains:

$$\mathcal{L}[e^{-at} f(t)] = F(s+a) \quad \text{and} \quad \mathcal{L}[f(t-a) \cdot u(t-a)] = e^{-as} F(s), \quad a \ge 0$$

6. LT of Derivatives:

$$\mathcal{L}[f'(t)] = sF(s) - f(0), \quad \mathcal{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

7. LT of Integrals: $\mathcal{L}[\int_0^t f(t)dt] = \frac{1}{s} F(s)$

8. Others: $\begin{cases} \mathcal{L}[tf(t)] = -\frac{d}{ds} F(s) \quad \Rightarrow \quad \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) \\ \mathcal{L}[\frac{1}{t} f(t)] = \int_s^\infty F(s) ds \\ \mathcal{L}[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t) dt \quad \text{for } f(t+p) = f(t), \forall t \end{cases}$

$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, $\det \mathbf{A} = ad - bc$

Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

$ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$

$\cosh ax = \frac{1}{2}\left(e^{ax} + e^{-ax}\right), \quad \sinh ax = \frac{1}{2}\left(e^{ax} - e^{-ax}\right)$

$2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$
$2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$
$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$
$2\sin A \cdot \sin B = -\cos(A+B) + \cos(A-B)$

- General solution of the 1st-order ODE, $y' + p(x)\, y = r(x)$ is written as
  $y(x) = e^{-h(x)}\left[\int e^{h(x)} r(x) dx + C\right]$ where $h(x) = \int p(x) dx$.

Differentiation

$(cu)' = cu' \qquad (c \text{ constant})$

$(u + v)' = u' + v'$

$(uv)' = u'v + uv'$

$\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

$\dfrac{du}{dx} = \dfrac{du}{dy} \cdot \dfrac{dy}{dx} \qquad$ (Chain rule)

$(x^n)' = nx^{n-1}$

$(e^x)' = e^x$

$(e^{ax})' = ae^{ax}$

$(a^x)' = a^x \ln a$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$(\ln x)' = \dfrac{1}{x}$

$(\log_a x)' = \dfrac{\log_a e}{x}$

$(\arcsin x)' = \dfrac{1}{\sqrt{1 - x^2}}$

$(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}$

$(\arctan x)' = \dfrac{1}{1 + x^2}$

$(\operatorname{arccot} x)' = -\dfrac{1}{1 + x^2}$

Integration

$\int uv' \, dx = uv - \int u'v \, dx$ (by parts)

$\int x^n \, dx = \dfrac{x^{n+1}}{n + 1} + c \qquad (n \neq -1)$

$\int \dfrac{1}{x} \, dx = \ln |x| + c$

$\int e^{ax} \, dx = \dfrac{1}{a} e^{ax} + c$

$\int \sin x \, dx = -\cos x + c$

$\int \cos x \, dx = \sin x + c$

$\int \tan x \, dx = -\ln |\cos x| + c$

$\int \cot x \, dx = \ln |\sin x| + c$

$\int \sec x \, dx = \ln |\sec x + \tan x| + c$

$\int \csc x \, dx = \ln |\csc x - \cot x| + c$

$\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a} \arctan \dfrac{x}{a} + c$

$\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \arcsin \dfrac{x}{a} + c$

$\int \dfrac{dx}{\sqrt{x^2 + a^2}} = \operatorname{arcsinh} \dfrac{x}{a} + c$

$\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \operatorname{arccosh} \dfrac{x}{a} + c$

$\int \sin^2 x \, dx = \tfrac{1}{2}x - \tfrac{1}{4} \sin 2x + c$

$\int \cos^2 x \, dx = \tfrac{1}{2}x + \tfrac{1}{4} \sin 2x + c$

$\int \tan^2 x \, dx = \tan x - x + c$

$\int \cot^2 x \, dx = -\cot x - x + c$

$\int \ln x \, dx = x \ln x - x + c$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c$$

$\int x \sin x dx = -x \cos x + \sin x + C$

$\int x \cos x dx = x \sin x + \cos x + C$

$\int \cos^3 x \sin x dx = -\tfrac{1}{4} \cos^4 x + C$

$\int \sin^3 x \cos x dx = \tfrac{1}{4} \sin^4 x + C$

$$I_n \triangleq \int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} \sin^n x \, dx$$
$$\Rightarrow I_n = \tfrac{n-1}{n} I_{n-2}, \quad I_0 = \tfrac{\pi}{2} \text{ and } I_1 = 1$$

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s) \, ds$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \oint_C (F_1 dx + F_2 dy)$$

$$\iiint_T \text{div } \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) \, du \, dv$$

$$x(t) = x(t+T), \quad \omega_0 = 2\pi / T$$
$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right)$$

where
$$\begin{cases} a_0 = \dfrac{1}{T} \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \, dt \\ a_n = \dfrac{2}{T} \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\omega_0 t \, dt \\ b_n = \dfrac{2}{T} \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t \, dt \end{cases}$$